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A unified framework for spline estimators

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Abstract

This article develops a unified framework to study the (asymptotic) properties of (periodic) spline based estimators, that is of regression, penalized and smoothing splines. We obtain an explicit form of the Demmler-Reinsch basis of general degree in terms of exponential splines and corresponding eigenvalues by applying Fourier techniques to periodic smoothers. This allows to derive exact expressions for the equivalent kernels of all spline estimators and get insights into the local and global asymptotic behavior of these estimators.
Key words and phrases: B-splines; Equivalent kernels; Euler-Frobenius polynomials; Exponential splines; Demmler-Reinsch basis.

1 Introduction

Spline based estimators have a long history in nonparametric regression. The idea of smoothing splines traces back to Whittaker (1923) and has been developed further among many others by Schoenberg (1964) and Reinsch (1967), as well as by Wahba (1975) who popularized smoothing splines in statistics. Another spline based technique is regression (or least-squares) splines introduced in works of Hartley (1961) and Hudson (1966),

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among others. Over last two decades penalized (or low-rank) splines have become increasingly popular, see Ruppert et al. (2003). Penalized splines combine projection onto a low dimensional spline space (as by regression splines) with the roughness penalty (as by smoothing splines) and circumvent herewith certain practical disadvantages of smoothing and regression splines.

Investigation of the statistical properties of these three spline based estimators has been based on very different approaches. Agarwal and Studden (1980) and Zhou et al. (1998) studied both local and global asymptotics of the regression spline estimators making use of the results of Barrow and Smith (1978), who found a sharp estimate of the error for the best L_2 approximation of a smooth function by a splines set. Another approach have taken Huang and Studden (1993), who derived an equivalent kernel in the special case of cubic regression splines.

For smoothing splines, Fourier techniques (Rice and Rosenblatt, 1981, 1983; Cogburn and Davis, 1974), the reproducing kernel Hilbert spaces framework (Craven and Wahba, 1978) and the asymptotic correspondance of the smoothing spline minimization problem to a certain boundary value problem (Utreras, 1983) have been employed to obtain L_2 error bounds for the smoothing spline estimators. To understand the local properties of smoothing spline estimators, asymptotic equivalent kernels have been extensively studied. First, Cogburn and Davis (1974) obtained an asymptotic equivalent kernel for smoothing splines on the real line, using Fourier techniques. Messer and Goldstein (1993) and Thomas-Agnan (1996) extended this kernel to the case of a bounded interval. Later, Eggermont and LaRiccia (2006) refined these two results.

The asymptotic properties of penalized spline estimators have got attention only recently. It has been discussed in Claeskens et al. (2009) that depending on the number of knots taken, penalized splines have asymptotic behavior similar either to regression or to smoothing splines. Kauermann et al. (2009) studied in more detail the "small" number of knots scenario in the generalized regression context. Recently, Wang et al. (2011) have shown that in the asymptotic scenario with the "large" number of knots, the equivalent kernel for penalized splines is asymptotically equivalent to that of smoothing splines. All these works used mixed approaches, combining techniques for regression and spline estimators, depending on the asymptotic scenario.

In this article we study in a unified framework all spline based estimators: regression, penalized and smoothing splines. Obtained new explicit expression for the Demmler-Reinsch basis for periodic splines allows not only to obtain the L_2 risk, but also to derive exact equivalent kernels for all spline estimators on $\mathbb{R}$, not available before even for smoothing and regression splines. This delivers interesting insights into the local asymptotic behavior of the spline estimators.

The paper is organized as follows. Section 2 introduces necessary concepts and notations. Section 3 gives the Demmler-Reinsch basis, its eigenvalues and the Fourier coefficients of spline estimators. In Section 4 the L_2 risk for periodic spline estimators is given, while in Section 5 equivalent kernels are derived. Local asymptotics of spline estimators is discussed in Section 6 and Section 7 concludes the paper. All proofs are given in the Appendix.

2 Preliminaries and notations

Consider a nonparametric regression model for the data pairs (y_i, x_i), $i = 1, \dots, N$,

$$y_i = f(x_i) + \epsilon_i, \tag{1}$$

with the standard assumptions on the random errors, that is $E(\epsilon_i) = 0$ and $E(\epsilon_i \epsilon_j) = \sigma^2 \delta_{ij}$ with $\sigma^2 > 0$ and δ_{ij} as the Kronecker delta. Let the data to be equally spaced on the $[0,1]$ interval, i.e. $x_i = i/N$, $i = 1, \dots, N$. The unknown regression function f is assumed to be a periodic function with period 1. More precisely, $f \in \mathcal{P}_{p+1} = \{f : f \in C^{p+1}(\mathbb{R}),\ f^{(j)}(0+l) = f^{(j)}(1+l),\ l \in \mathbb{Z},\ j = 0, \dots, p\}$. To estimate $f \in \mathcal{P}_{p+1}$ with splines, define first the partition of $[0,1]$ into K, $K \leq N$, equidistant intervals $\underline{\tau}_K = \{0 = \tau_0 < \tau_1 < \dots < \tau_{K-1} < \tau_K = 1\}$ with $\tau_i = i/K$, $i = 0, \dots, K$. Without loss of generality, assume K to be even and $M = N/K$ (the number of observations in each interval $[\tau_i, \tau_{i+1})$) to be an integer. Further, the periodic spline space $S_{\mathtt{per}}(p; \underline{\tau}_K)$ of degree $p > 0$ based on $\underline{\tau}_K$ consists of functions s, such that $s \in C^{p-1}[0,1]$, s is a degree p polynomial on each $[\tau_i, \tau_{i+1})$, $i = 0, \dots, K-1$ and $s^{(j)}(0) = s^{(j)}(1)$, $j = 0, \dots, p-1$. The periodic spline estimator $\widehat{f}$ of f is found as the solution to

$$\min_{s \in S_{\mathtt{per}}(p; \underline{\tau}_K)} \left[\frac{1}{N} \sum_{i=1}^{N} \{Y_i - s(x_i)\}^2 + \lambda \int_0^1 \{s(x)^{(q)}\}^2 dx \right], \tag{2}$$

$\lambda \geq 0,\ 0 < q \leq p$. For $K = N$ and $p = 2q-1$ the solution to (2) is the periodic smoothing spline estimator (Wahba, 1975). If $\lambda = 0$ and $K \ll N$, (2) results in the periodic regression spline estimator (Zhou et al., 1998) and a general estimator with $K < N$, $p+1 > q > 0$ and $\lambda > 0$ is the so-called low-rank or penalized spline estimator (Claeskens et al., 2009). Let $B_c(x)$ be a cardinal B-spline of degree p (for the definition see Schumaker, 1981). Here and subsequently, we suppress in the notation the degree of the spline p. Then the periodic B-spline basis for $S_{\mathtt{per}}(p; \underline{\tau}_K)$ can be defined as

$$B_i(x) = \sum_{l=-\infty}^{\infty} B_c\left\{K(x + l - i/K)\right\},\ i = 1, \dots, K,\ \ x \in [0,1].$$

The Fourier transform of $B_c(x)$ is known to be $\mathrm{sinc}(x)^{p+1}$, where $\mathrm{sinc}(x)$ denotes $\sin(x)/x$, implying that the Fourier series of a periodic B-spline is given by

$$B_i(x) = \sum_{l=-\infty}^{\infty} \mathrm{sinc}(\pi l/K)^{p+1} \exp\{2\pi \mathtt{i} l(x - i/K)\}, \tag{3}$$

$i = 1, \ldots, K$. In the following, we make use of the relationship

$$\sum_{l=-\infty}^{\infty} \mathrm{sinc}\{\pi(z+l)\}^{p+1} = Q_{p-1}(z), \tag{4}$$

where $Q_{p-1}(z) = 1$ for $z \in \mathbb{Z}$, while for $z \notin \mathbb{Z}$ this is the polynomial of $\cos(\pi z)$ of degree $(p-1)$ defined recursively via

$$Q_j(z) = \cos(\pi z) Q_{j-1}(z) + \frac{1 - \cos(\pi z)^2}{j+1} \frac{dQ_{j-1}(z)}{d\cos(\pi z)}, \tag{5}$$

$j = 1, \ldots, p-1$, with $Q_0(z) = 1$ for p odd. If p is even, the recursive formula (5) is applied with $Q_0(z) = \cos(\pi z)$. The exact formulas for some first Q-polynomials for odd and even p, as well as the proof of (4) are given in the appendix. In a somewhat different context similar polynomials have been discussed in Gautschi (1971) who studied attenuation factors in the approximation of the Fourier coefficients of f available on a grid of N values $f(i/N)$. In fact, Q-polynomials are closely connected to the well-known Euler-Frobenius polynomials $\Pi_p(\cdot)$, see Schoenberg (1973). For odd p this relationship is simple:

$$Q_{p-1}(z) = \exp\{\mathtt{i} z\pi(p-1)\}\Pi_p\{\exp(-2\mathtt{i}\pi z)\}/p!.$$

For p even it can be expressed as

$$\begin{aligned} Q_{p-1}(z) &= \exp\{\pi \mathtt{i} z(p-1)/2\} \cos(\pi z/2)^{p+1} \Pi_p \{\exp(-\pi \mathtt{i} z)\}/p! \\ &- (-1)^{p/2} \mathtt{i} \exp\{\pi \mathtt{i} z(p-1)/2\} \sin(\pi z/2)^{p+1} \Pi_p \{-\exp(-\pi i z)\}/p!. \end{aligned}$$

In Figure 1, several first polynomials $Q_{p-1}(z)$ for odd and even p are shown. Further, we introduce the exponential splines, see Schoenberg (1973), which are defined in terms of Euler-Frobenius polynomials as

$$\Phi_p(t, z) = z^{\lfloor t \rfloor} \left(1 - z^{-1}\right)^p \sum_{j=0}^{p} \binom{p}{j} \frac{\{t\}^{p-j} \, \Pi_j(z)}{p! \, (z-1)^j}, \tag{6}$$

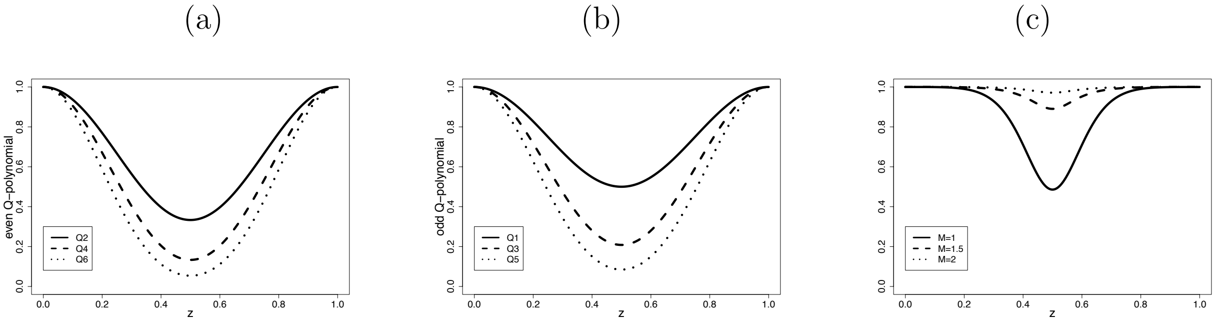

Figure 1: (a) Q_{p-1} polynomials for p odd, (b) Q_{p-1} polynomials for p even, (c) $Q_6(z)/Q_{3,M}(z)$ for different M.

$z \neq 0,\ z \neq 1$, where $\{t\}$ denotes the fractional part of t and $\lfloor t \rfloor$ is the largest integer not greater than t. With the convention $0^0 = 1$, one can also define $\Phi_p(t,1) = 1$. Note that

$$\begin{aligned}\Phi_p\{t, \exp(2\pi \mathtt{i} z)\} &= \frac{\exp(2\pi \mathtt{i} z t)}{\exp\{\pi \mathtt{i} z(p+1)\}} \\ &\times \sum_{l=-\infty}^{\infty} (-1)^{l(p+1)} \operatorname{sinc}\{\pi(z+l)\}^{p+1} \exp(2\pi \mathtt{i} l t). \qquad (7)\end{aligned}$$

This equation, as well as connection of Q-polynomials to Euler-Frobenius polynomials are discussed in more detail in the appendix. Finally, we define

$$Q_{p,M}(z) = \frac{1}{N} \sum_{i=1}^{N} \left|\Phi_p\left\{i/M + (p+1)/2, \exp(-2\pi \mathtt{i} z)\right\}\right|^2. \qquad (8)$$

For $M = 1$ we find $Q_{p,1}(z) = Q^2_{p-1}(z)$, since $\Phi\left\{(p+1)/2, \exp(2\pi \mathtt{i} z)\right\} = Q_{p-1}(z)$ from (7). If $M = N/K > 1$, then $Q_{p,M}(z)$ varies between $Q_{2p}(z)$ and $Q^2_{p-1}(z)$, depending on M. For example, for $p = 1$ and $p = 3$ one obtains

$$\begin{aligned}Q_{1,M}(z) &= Q_2(z) + \frac{2\sin(\pi z)^2}{3M^2}, \\ Q_{3,M}(z) &= Q_6(z) + \frac{2\{3\sin(\pi z)^4 - 2\sin(\pi z)^6\}}{135M^4} + \frac{8\sin(\pi z)^6}{189M^6},\end{aligned}$$

and in general $Q_{p,M}(z) = Q_{2p} + O(M^{-p-1})$, so that for growing M polynomial $Q_{p,M}(z)$ converges exponentially to $Q_{2p}(z)$, as the right hand side plot of Figure 1 demonstrates for $Q_6(z)/Q_{3,M}(z)$.

3 Demmler-Reinsch basis for periodic splines

We start by stating the lemma, which gives the explicit expression for the complex-valued Demmler and Reinsch (1975) basis for the periodic spline space $S_{\mathtt{per}}(p; \underline{\tau}_K)$.

Lemma 1 *For $x \in \mathbb{R}$ functions*

$$\phi_i\,(x) = \frac{\Phi_p\{Kx + (p+1)/2, \exp(-2\pi \mathtt{i} i/K)\}}{\sqrt{Q_{p,M}(i/K)}} \tag{9}$$

$i = 1, \ldots, K$ form the complex-valued Demmler-Reinsch basis in $S_{\mathtt{per}}(p; \underline{\tau}_K)$, i.e. it holds

$$\frac{1}{N}\sum_{l=1}^{N} \phi_i(l/N)\overline{\phi_j(l/N)} \;=\; \delta_{i,j} \tag{10}$$

$$\int_0^1 \phi_i^{(q)}(x)\overline{\phi_j^{(q)}(x)}dx \;=\; \mu_i\delta_{i,j}, \tag{11}$$

$i, j = 1, \ldots, K$, with $\delta_{i,j}$ as the Kronecker's delta and the eigenvalues

$$\mu_i = (2\pi i)^{2q}\mathrm{sinc}(\pi i/K)^{2q}\frac{Q_{2p-2q}(i/K)}{Q_{p,M}(i/K)}. \tag{12}$$

Remarks

1. $\int_0^1 \phi_i(x)\overline{\phi_j(x)}dx = \delta_{ij}Q_{2p}(i/K)/Q_{p,M}(i/K)$.

2. Basis functions $\phi_i(x)$ is the scaled discrete Fourier transform of periodic B-splines, i.e. $\phi_i(x)K\sqrt{Q_{p,M}(i/K)} = \sum_{l=1}^{K} B_l(x)\exp(-2\pi \mathtt{i} il/K)$. A similar basis for $N = K$ has been considered in Lee et al. (1992), who studied Fourier coefficients $\widetilde{f}_i$, given in Theorem 1 of Section 4.

3. Since $\phi_i(x)$ is the scaled discrete Fourier transform of a real-valued B-spline functions and $Q_{p,M}(z)$ is a symmetric, positive function by definition, it holds that $\phi_i(x) = \overline{\phi_{K-i}(x)}$ and $\mu_i = \mu_{K-i}$.

4. Even though the Demmler-Reinsch basis for periodic smoothing splines has been in employed in Cogburn and Davis (1974) and Craven and Wahba (1978), no explicit expressions for ϕ_i and μ_i have been derived there. For $K = N$ and $p = 2q - 1$, $\mu_i = (2\pi \mathtt{i})^{2q}\text{sinc}(\pi i/K)^{2q} Q_{2q-2}(i/K)^{-1}$ and at the data points l/N, the Demmler-Reinsh basis reduces to $\phi_i(l/N) = \exp(-2\pi \mathtt{i} il)$.

Thus, any $s(x) \in S_{\mathtt{per}}(p; \underline{\tau}_K)$ can be represented as $s(x) = \sum_{i=1}^{K} \beta_i \phi_i(x)$ and the solution to (2) results in $\widehat{f}(x) = \sum_{i=1}^{K} \widehat{\beta}_i \phi_i(x)$ with $\widehat{\beta}_i = (1 + \lambda\mu_i)^{-1}\widehat{y}_i$, where $\widehat{y}_i = N^{-1} \sum_{l=1}^{N} y_l \; \overline{\phi_i(l/N)}$. Using (7) and the definition of the complex-valued Demmler-Reinsch basis (9), we can write

$$\begin{aligned} \phi_i(x) &= \frac{1}{\sqrt{Q_{p,M}(i/K)}} \\ &\times \sum_{l=-\infty}^{\infty} \text{sinc}\{\pi(i/K + l)\}^{p+1} \exp\{-2\pi \mathtt{i} x(i + lK)\}, \end{aligned} \tag{13}$$

so that

$$\widehat{f}(x) = \frac{1}{\sqrt{Q_{p,M}(i/K)}} \sum_{l=-\infty}^{\infty} \sum_{i=1}^{K} \widehat{\beta}_i \, \text{sinc}\{\pi(i/K + l)\}^{p+1} \exp\{-2\pi \mathtt{i} x(i + lK)\}.$$

Since $\widehat{\beta}_i = \widehat{\beta}_{i+lK}$ and $Q_{p,M}(i/K) = Q_{p,M}(i/K + l)$, the Fourier coefficients of the periodic spline estimator are given by

$$c_{i+lK} = \frac{\text{sinc}\{\pi(i/K + l)\}^{p+1}}{\sqrt{Q_{p,M}(i/K)}} \widehat{\beta}_i = \frac{\text{sinc}\{\pi(i/K + l)\}^{p+1} \widehat{y}_i}{\sqrt{Q_{p,M}(i/K)}(1 + \lambda\mu_i)}, \tag{14}$$

where the c_l satisfy $\widehat{f}(x) = \sum_{l=-\infty}^{\infty} c_l \exp(-2\pi \mathtt{i} lx)$. From (14), one can immediately obtain the Fourier coefficients for both extreme cases: periodic smoothing and regression splines. In particular, for $K = N$ and $p = 2q - 1$ (periodic smoothing spline)

$$c_{i+lN} = \frac{\mathrm{sinc}\{\pi(i/N+l)\}^{2q}\breve{y}_i}{Q_{2q-2}(i/N) + \lambda(2\pi i)^{2q}\mathrm{sinc}(\pi i/N)^{2q}},$$

with $\breve{y}_i = N^{-1}\sum_{l=1}^{N}\exp(2\pi \mathtt{i} il/N)y_l$ and for $\lambda = 0,\ K \ll N$ (periodic regression spline)

$$c_{i+lK} = \frac{\mathrm{sinc}\{\pi(i/K+l)\}^{p+1}\widehat{y}_i}{\sqrt{Q_{p,M}(i/K)}} = \frac{\mathrm{sinc}\{\pi(i/K+l)\}^{p+1}\widehat{y}_i}{\sqrt{Q_{2p}(i/K) + O(M^{-p-1})}}.$$

4 L_2 risk for periodic spline estimators

The L_2 risk of a spline estimator can be decomposed as

$$\begin{aligned}\mathrm{R}(\widehat{f}, f) &= \int_0^1 E\{\widehat{f}(x) - f(x)\}^2 dx = \int_0^1 \mathrm{Var}\{\widehat{f}(x)\}dx \\ &+ \int_0^1 [E\{\widehat{f}(x)\} - s_{\mathtt{p}}(x)]^2 dx + \int_0^1 \{s_{\mathtt{p}}(x) - f(x)\}^2 dx,\end{aligned}$$

where $s_{\mathtt{p}}$ is the best $L_2[0,1]$ approximation of $f \in \mathcal{P}_{p+1}$ by $S_{\mathtt{per}}(p; \underline{\tau}_K)$. Thus, the $\mathrm{R}(\widehat{f}, f)$ consists of three summands: the integrated variance, the integrated squared shrinkage bias and the integrated squared approximation bias. The shrinkage bias appears due to the penalization involved in (2) and vanishes for regression splines ($\lambda = 0$). The approximation bias is the error due to approximation of a continuous function f by a spline. The sharp asymptotic behavior of the integrated squared approximation bias has been proved in Barrow and Smith (1978). In particular, they have shown that

$$\begin{aligned}\lim_{K\to\infty} K^{2p+2}\int_0^1 \{s_{\mathtt{p}}(x) - f(x)\}^2 dx &= \frac{|b_{2p+2}|}{(2p+2)!} \\ &\times \left(\int_0^1 |f^{p+1}(x)|^{1/(p+1.5)}dx\right)^{(2p+3)}, \qquad (15)\end{aligned}$$

with b_{2p+2} as the $(2p+2)$th Bernoulli number. The following theorem gives exact expressions for the integrated variance and the integrated squared shrinkage bias.

Theorem 1 *Let the model (1) hold, $f \in \mathcal{P}_{p+1}$ and $\widehat{f}(x) \in S_{\mathtt{per}}(p; \underline{\tau}_K)$ be the solution to (2) with $x_i = i/N$, $i = 1, \ldots, N$ and $\underline{\tau}_K = \{i/K\}_{i=0}^K$. Then, the integrated variance and the integrated squared shrinkage bias of $\widehat{f}(x)$ are given by*

$$\int_0^1 \mathrm{Var}\{\widehat{f}(x)\}dx = \frac{\sigma^2}{N}\sum_{i=1}^{K}\frac{Q_{2p}(i/K)}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2} \tag{16}$$

$$\int_0^1 [\mathrm{E}\{\widehat{f}(x)\} - s_{\mathrm{p}}(x)]^2 dx = \sum_{i=1}^{K}\frac{Q_{2p}(i/K)(\lambda\mu_i)^2|\widetilde{f}_i|^2}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2}\left|1 - \frac{\widehat{f}_i - \widetilde{f}_i}{\widetilde{f}_i\lambda\mu_i}\right|^2, \tag{17}$$

with $\widetilde{f}_i = \sqrt{Q_{p,M}(i/K)/Q_{2p}(i/K)}\int_0^1 f(x)\phi_i(x)dx$ *and*
$\widehat{f}_i = \sqrt{Q_{p,M}(i/K)/Q_{2p}(i/K)}N^{-1}\sum_{l=1}^{N} f(l/N)\phi_i(l/N)$.

Note that the ratio $Q_{2p}(z)/Q_{p,M}(z)$ is bounded and varies between $0 < Q_{2p}(z)/Q_{p-1}^2(z) < 1$ for $M = 1$ and 1 for $M \to \infty$, see the discussion at the end of Section 2 and the right hand side plot in Figure 1.

From the equations (16) and (17) it is clear that the asymptotic behavior of spline based estimators depends on $\lambda\mu_{K/2} = \lambda(2K)^{2q}Q_{2p-2q}(1/2)/Q_{p,M}(1/2)$, similar to the results in Claeskens et al. (2009). From (15), (16) and (17), one can find the asymptotic orders of the components of $\mathrm{R}(\widehat{f}, f)$ in two asymptotic scenarios.

Corollary 1 *Let the assumptions of Theorem 1 hold. Then for $p \geq 2q - 1$*

$$\mathrm{R}(\widehat{f}, f) = \begin{cases} O\left(KN^{-1}\right) + O\left(\lambda^2\right) + O(K^{-2p-2}), & \text{for } \lambda\mu_{K/2} = O(1), \\ O\left(\lambda^{-1/(2q)}N^{-1}\right) + O\left(\lambda^2\right) + O(K^{-2p-2}), & \text{for } \lambda\mu_{K/2} \to \infty, \end{cases}$$

so that for $\lambda\mu_{K/2} = O(1)$ and $K = cN^{1/(2p+3)}$, $\lambda = O(N^{-\nu})$, $\nu \in [(p+1)/(2p+3), 1]$ imply $\mathrm{R}(\widehat{f}, f) = O\left\{N^{-(2p+2)/(2p+3)}\right\}$. For $\lambda\mu_{K/2} \to \infty$ and $\lambda = O\left\{N^{-2q/(4q+1)}\right\}$ with $\lambda^{1/(2q)}N \to \infty$, $K = cN^{\varsigma}$, $\varsigma \in [1/(4q+1), 1]$ it follows $\mathrm{R}(\widehat{f}, f) = O\left\{N^{-4q/(4q+1)}\right\}$. Here c denotes a generic positive constant.

Corollary 1 states that depending on $\lambda\mu_{K/2}$, and thus on the number of knots K taken, the asymptotic scenario of (periodic) spline based estimators is either similar to the (periodic) regression spline asymptotics or to the (periodic) smoothing spline asymptotics. For small $K = cN^{1/(2p+3)}$ with $\lambda\mu_{K/2} = O(1)$ the convergence rate of the estimator $N^{-(2p+2)/(2p+3)}$ is the same as that for the regression splines (see Zhou et al., 1998) and λ is, in fact, non-identifiable, i.e. can not be estimated consistently. Once more knots are taken so that $\lambda\mu_{K/2} \to \infty$, the smoothing parameter λ controls the fit and the convergence rate is $N^{-4q/(4q+1)}$, as was found for periodic smoothing spline estimators by Wahba (1975). In this scenario K is non-identifiable, meaning that taking any K satisfying $K = cN^{\varsigma}$, $\varsigma \in [1/(4q+1), 1]$ has no influence on $\mathrm{R}(\widehat{f}, f)$. Apparently, the choice of p and q is important for the convergence rate in each scenario. Taking $p > 2q - 1$ leads to a faster convergence rate in the "small" number of knots scenario, while for $p = 2q - 1$ the convergence rate in both scenarios is the same.

5 Equivalent kernels on $\mathbb{R}$

Using (9), one can write the solution to (2) as $\widehat{f}(x) = N^{-1}\sum_{l=1}^{N} W(x, l/N)Y_l$, where

$$W(x,t) = \sum_{i=1}^{K} \frac{\overline{\phi_i(t)}}{1+\lambda\mu_i}\phi_i(x),$$

which is obviously in $S_{\mathtt{per}}(p; \underline{\tau}_K)$ for a fixed t (or fixed x). The space of periodic splines $S_{\mathtt{per}}(p; \underline{\tau}_K)$ is a finite dimensional Hilbert space and thus has a reproducing kernel. With respect to the inner product $< f, g >= N^{-1}\sum_{i=1}^{N} f(i/N)g(i/N) + \lambda\int_0^1 f^{(q)}(x)g^{(q)}(x)dx$ for any $N \geq K$ the reproducing kernel for $S_{\mathtt{per}}(p; \underline{\tau}_K)$ is $W(x, t)$.

In this section we consider $W(x, t)$ for $x, t \in \mathbb{R}$ and define additionally

$$\mathcal{W}(x,t) = \int_0^K \frac{\phi(u,x)\overline{\phi(u,t)}}{1+\lambda\mu(u)}du,$$

where $\mu(u) = (2\pi u)^{2q}\text{sinc}(\pi u/K)^{2q} Q_{2p-2q}(u/K)/Q_{p,M}(u/K)$ and $\phi(u,x) = \Phi_p\{Kx+(p+1)/2, \exp(-2\pi \mathtt{i} u/K)\}/\sqrt{Q_{p,M}(u/K)}$. In fact, $W(x,t)$ can be obtained by "folding-back" $\mathcal{W}(x,t)$, that is

$$W(x,t) = \sum_{l=-\infty}^{\infty} \mathcal{W}(x, t+l). \tag{18}$$

This can be proved by showing that the Fourier coefficients of both functions coincide. The Fourier coefficients of $W(x,t)$ as a function of t at a fixed x can be found from

$$W(x,t) = \sum_{l=-\infty}^{\infty} \sum_{i=1}^{K} \frac{\text{sinc}\{\pi(i/K+l)\}^{p+1}\phi_i(x)}{\sqrt{Q_{p,M}(i/K)}(1+\lambda\mu_i)} \exp\{2\pi \mathtt{i} t(i+lK)\},$$

simliar to the Fourier coefficients c_{i+lK} of $\widehat{f}(x)$, as given in Section 3. Since $Q_{p,M}(i/K) = Q_{p,M}(i/K+l)$, $\mu_i = \mu_{i+lK}$ and $\phi_i(x) = \phi_{i+lK}(x)$, we obtain

$$a_l(x) = \frac{\text{sinc}\{\pi(l/K)\}^{p+1}\phi_l(x)}{\sqrt{Q_{p,M}(l/K)}(1+\lambda\mu_l)},\ l \in \mathbb{Z}, \tag{19}$$

for $W(x,t) = \sum_{l=-\infty}^{\infty} a_l(x)\exp(2\pi \mathtt{i} l t)$. From the Poisson summation formula

$$\int_0^1 \sum_{j=-\infty}^{\infty} \mathcal{W}(x,t+j)\exp(-2\pi \mathtt{i} t l)dt = \int_{-\infty}^{\infty} \mathcal{W}(x,t)\exp(-2\pi i t l)dt \tag{20}$$

follows the equality of lth Fourier coefficients of $\sum_{j=-\infty}^{\infty}\mathcal{W}(x,t+j)$ and of the Fourier transform of $\mathcal{W}(x,t)$. Applying the Poisson summation formula again we obtain

$$\begin{aligned} \mathcal{W}(x,t) &= \int_0^K \sum_{l=-\infty}^{\infty} \frac{\text{sinc}\{\pi(u/K+l)\}^{p+1}\phi(u,x)}{\sqrt{Q_{p,M}(u/K)}\{1+\lambda\mu(u)\}} \exp\{2\pi \mathtt{i} t(u+lK)\}du \\ &= \int_{-\infty}^{\infty} \frac{\text{sinc}\{\pi(u/K)\}^{p+1}\phi(u,x)}{\sqrt{Q_{p,M}(u/K)}\{1+\lambda\mu(u)\}} \exp\{2\pi \mathtt{i} t u\}du. \end{aligned} \tag{21}$$

From (20), (21) and the inverse Fourier transform follows the equality of the Fourier coefficients of $\sum_{j=-\infty}^{\infty}\mathcal{W}(x,t+j)$ and $a_l(x)$ in (19), which proves (18). The next lemma gives exact expressions for $W(x,t)$ and $\mathcal{W}(x,t)$.

Lemma 2 *Let*

$$\begin{aligned} P_{2p}(u) &= \widetilde{\Pi}_{p,M}(u) \\ &+ (-1)^q \lambda K^{2q}(1-u)^{2q}\Pi_{2p-2q+1}(u)/(2p-2q+1)! \end{aligned} \tag{22}$$

be a polynomial of degree $2p$ *where* $\Pi_{2p-2q+1}(u)$ *is the Euler-Frobenius polynomial and* $\widetilde{\Pi}_{p,M}(u)$ *is a linear combination of certain Euler-Frobenius polynomials with its exact expression given in the proof. Let also* r_j, r_j^{-1}, $j = 1,\ldots,p$ *be the roots of* $P_{2p}(u)$ *with* $|r_j| < 1$. *Then, denoting* $P'_{2p}(r_j) = \partial P_{2p}(u)/\partial u|_{u=r_j}$, $d_{x,t} = \lfloor Kx - \{\frac{p+1}{2}\}\rfloor - \lfloor Kt - \{\frac{p+1}{2}\}\rfloor$ *and representing*

$$z^{p-d_{x,t}}\Phi_p\{Kx+(p+1)/2, z\}\Phi_p\{Kt+(p+1)/2, z^{-1}\} = \sum_{l=0}^{2p}\alpha_l(\{Kx\},\{Kt\})z^l$$

for some functions $\alpha_l(t_1,t_2)$ *and* $x,t \in \mathbb{R}$, *results in*

$$\begin{aligned} W(x,t) &= K\sum_{j=1}^{p}\sum_{l=0}^{2p}\frac{\alpha_l\left(\{Kx\},\{Kt\}\right)}{P'_{2p}(r_j)} \\ &\times \frac{r_j^{(d_{x,t}+l-1) \bmod K} + r_j^{K+2p-2-(d_{x,t}+l-1) \bmod K}}{\left(1-r_j^K\right)}, \end{aligned}$$

$$\mathcal{W}(x,t) = K\sum_{j=1}^{p}\sum_{l=0}^{2p}\frac{\alpha_l\left(\{Kx\},\{Kt\}\right)}{P'_{2p}(r_j)} r_j^{|d_{x,t}+l-1|+I_{\{d_{x,t}\le -l\}}(2p-2)}. \tag{23}$$

There is a closed form expression available for the roots of polynomial P_{2p} for degrees $p \le 4$. For these degrees both $W(x,t)$ and $\mathcal{W}(x,t)$ can be obtained explicitly. For larger degrees, approximate expression for the roots are given in Sobolev (2006). For $p = q = 1$ both functions have simple representations, for larger p and q they become much more involved. In particular, for $p = q = 1$ one finds $r_1 = 1 - (\sqrt{6/M^2+3+36\lambda K^2} - 3)/(6\lambda K^2 - 1 + 1/M^2)$, $P'_2(r_1) = \sqrt{(2/M^2+1+12\lambda K^2)/3}$, $\alpha_2(t_1,t_2) = \alpha_0(t_2,t_1) = t_1 - t_1 t_2$

and $\alpha_1(t_1, t_2) = 1 - \alpha_0(t_1, t_2) - \alpha_2(t_1, t_2)$.

Equivalent kernels obtained in Lemma 2 look very complicated, but they are exact and hold for any p, q, M, K and λ. Known equivalent kernels for smoothing splines on $\mathbb{R}$ correspond to $\mathcal{W}(x,t)$ with $K = N \to \infty$. Equivalent kernels for regession splines on $\mathbb{R}$ of Huang and Studden (1993) are also exact, but were obtained in terms of a certain (rather involved) linear combination of B-splines for $p = 3$ only. As Nychka (1995) notices, one does not need to know the exact form of the kernel in order to study the pointwise bias and variance of smoothing splines. Similarly, in the proof of our Theorem 3 for local asymptotics of general spline estimators, it turns out that only the following Lemma 3 is crucial.

Lemma 3 *For $x, t \in \mathbb{R}$, it holds for $m \le \min\{p, 2q-1\}$ that*

$$\int_{-\infty}^{\infty} (t-x)^m \, \mathcal{W}(x,t)\, dt = \int_0^1 (t-x)^m \, W(x,t)\, dt = 0,$$

while for $m = \min\{p+1, 2q\}$,

$$\begin{aligned}\int_{-\infty}^{\infty} (t-x)^m \, \mathcal{W}(x,t)\, dt &= \int_0^1 (t-x)^m \, W(x,t)\, dt \\ &= I_{\{p\le 2q-1\}} \left\{ \frac{2\mathcal{B}_{p+1}\left(\left\{\frac{p+1}{2}\right\}\right)}{N^{p+1}} - \frac{\mathcal{B}_{p+1}\left(\left\{Kx + \frac{p+1}{2}\right\}\right)}{K^{p+1}} \right\} - I_{\{p\ge 2q-1\}} \frac{\lambda\,(2q)!}{(-1)^q},\end{aligned}$$

with $\mathcal{B}_{p+1}(x)$ as a $(p+1)$-th degree Bernoulli polynomial and $I_{\{p\ge 2q-1\}}$ as an indicator function.

6 Local Asymptotics on $\mathbb{R}$

Let us define the variable $k_q = \lambda^{1/(2q)}\pi K$, which differs by a constant from $(\lambda\mu_{K/2})^{1/(2q)}$ and in the same way determines the type of the spline estimator. In particular, $k_q = 0$ corresponds to the regression spline estimator, $k_q = \lambda^{1/(2q)}\pi N \to \infty$ to the smoothing spline estimator and all intermediate values characterize penalized spline estimators. Before we present the results on the local asymptotics for spline estimators, let us introduce a bandwidth $h(k_q)$, which is universal for all spline estimator and is defined via

$$h(k_q)^{-1} = \int_0^K \frac{dx}{1+\lambda(\pi x)^{2q}} = \lambda^{-1/(2q)}\pi^{-1}\int_0^{k_q} \frac{dx}{1+x^{2q}}.$$

Bandwidth $h(k_q)$ is a smooth function of k_q with a (rather complicated) closed form expression available for each q. However, for our subsequent developments the following representation will be more suitable.

$$h(k_q)^{-1} = \lambda^{-1/(2q)}\pi^{-1}\begin{cases} k_q\, c_1, & k_q < 1 \\ \pi\, c_2, & k_q \geq 1, \end{cases} \tag{24}$$

with constants $c_1 = c_1(k_q) = {}_2\mathcal{F}_1[\{1, 1/(2q)\}; \{1+1/(2q)\}, -k_q^{2q}]$ and $c_2 = c_2(k_q) = \widetilde{c}_2 - \pi^{-1}{k_q}^{1-2q}{}_2\mathcal{F}_1[\{1, 1-1/(2q)\}; \{2-1/(2q)\}, -k_q^{-2q}]/(2q-1)$, where $\widetilde{c}_2 = \pi^{-1}\mathrm{sinc}\{\pi/(2q)\}^{-1}$ is independent of k_q and ${}_2\mathcal{F}_1$ denotes the hypergeometric series (see Abramowitz and Stegun, 1972). Both $c_1(k_q)$ and $c_2(k_q)$ are convergent and vary slow with k_q, namely $c_1(k_q) \in (\pi/4, 1]$ and $c_2(k_q) \in (1/4, 1/2]$. Note that for the regression spline estimators ($k_q = 0$) the inverse bandwidth $h(0)^{-1} = Kc_1(0) = K$ and for the smoothing spline estimators ($k_q \to \infty$) the inverse bandwidth $h(\infty)^{-1} = \lambda^{-1/(2q)}\widetilde{c}_2$.

With this, we can define the equivalent kernel $\mathcal{K}(x,t)$ for the spline estimator $\widehat{f}(x)$, $x, t \in \mathbb{R}$, to be such that

$$\mathcal{W}(x,t) = \frac{1}{h(k_q)}\mathcal{K}\left(\frac{x}{h(k_q)}, \frac{t}{h(k_q)}\right).$$

It holds the following result.

Theorem 2 *The equivalent kernel for spline estimators on $\mathbb{R}$ for $p = 2q-1$ is given by*

$$\begin{cases} c_1\mathcal{K}(c_1x, c_1t) = \mathcal{K}_{rs}(x,t) - k_q^{2q}\mathcal{K}_1(x,t), & k_q < 1 \\ c_2\mathcal{K}(c_2x, c_2t) = \mathcal{K}_{ss}(t-x) + k_q^{1-2q}\mathcal{K}_2(x,t), & k_q \geq 1, \end{cases}$$

where $\mathcal{K}_1(x,t)$ and $\mathcal{K}_2(x,t)$ are bounded functions given in the proof, $\mathcal{K}_{rs}(x,t)$ is the re-

gression spline equivalent kernel

$$\mathcal{K}_{rs}(x,t) = \sum_{j=1}^{p}\sum_{l=0}^{2p} \frac{\alpha_l\left(\{x\},\{t\}\right)}{P'_{2p}(r_j)}\, r_j^{|d_{x,t}+l-1|+I_{\{d_{x,t}\le -l\}}(2p-2)},$$

with α_l, r_j, $d_{x,t}$, $P_{2p}(u) = \Pi_{2p}(u)$ *defined in Lemma 2 and* $\mathcal{K}_{ss}(x,t)$ *is the asymptotic smoothing spline equivalent kernel (i.e. for* $N \to \infty$*)*

$$\mathcal{K}_{ss}(x,t) \;=\; \mathcal{K}_{ss}(x-t) = \sum_{j=0}^{q-1} \frac{\mathtt{i}\exp\left[\mathtt{i}\,|x-t|\exp\left\{\pi\mathtt{i}\,(2j+1)\,/(2q)\right\}\right]}{2q\exp\left\{\mathtt{i}\pi(2q-1)\,(2j+1)\,/(2q)\right\}}.$$

From Theorem 2 follows that $\lim_{k_q\to\infty} c_2\mathcal{K}(c_2x,c_2t) = \widetilde{c}_2^{-1}\mathcal{K}_{ss}\left\{(x-t)/\widetilde{c}_2\right\}$ and $\lim_{k_q\to 0} c_1\mathcal{K}(c_1x,c_1t) = \mathcal{K}_{\mathtt{rs}}(x,t)$, that is, $\mathcal{K}(x,t)$ varies smoothly between $\mathcal{K}_{rs}(x,t)$ and $\mathcal{K}_{ss}(x,t)$, scaled with appropriate constants. This is visualized in Figures 2 and 3. Figure 2 shows $\mathcal{K}_{rs}(x,t)$ for $p=1,3$ at $t=0$, $t=0.3$ and $t=0.5$, as well as $\mathcal{K}_{ss}(x-t)$ for $q=1,2$. Obviously, $\mathcal{K}_{rs}(x,t)$ is not translation invariant, in contrast to $\mathcal{K}_{ss}(x,t)$. Figure 3 depicts the penalized spline kernel $\mathcal{K}(x,t)$ for $M=5$ at $t=0$ and $t=0.3$ for different values of k_q and for $p=1,3$. As k_q grows, $\mathcal{K}(x,t)$ becomes more symmetric and for $k_q=5$ is already non-distinguishable from the smoothing spline kernel shown in grey.

Finally, we proof the following theorem for the pointwise bias and pointwise variance of the periodic spline estimator $\widehat{f}(x)$ in terms of the universal bandwidth $h(k_q)$.

Theorem 3 *Let the model (1) hold and* $\widehat{f}(x) \in S_{\mathtt{per}}(2q-1;\underline{\tau}_K)$ *be the solution to (2) with* $x_i = i/N$, $i=1,\ldots,N$ *and* $\underline{\tau}_K = \{i/K\}_{i=0}^{K}$*. Then for* $f \in \mathcal{P}_{p+1}$*, such that* $f^{(2q)} \in C^{0,\alpha}([0,1])$*, i.e.* $f^{(2q)}$ *is Hölder continuous with* $|f^{(2q)}(x)-f^{(2q)}(t)| \le L|x-t|^\alpha$, $\forall x,t \in [0,1]$ *and* $\alpha \in (0,1]$*, it holds at any* $x \in [0,1]$

$$\mathrm{E}\left\{\widehat{f}(x)\right\} - f(x) = -h(k_q)^{2q}\frac{f^{(2q)}(x)}{(2q)!}C(k_q,x) + o\left\{h(k_q)^{2q}\right\}$$

$$\mathrm{Var}\left\{\widehat{f}(x)\right\} = \frac{\sigma^2}{Nh(k_q)}\int_{-\infty}^{\infty}\mathcal{K}^2\left\{x/h(k_q),t\right\}dt + o\left\{N^{-1}h(k_q)^{-1}\right\},$$

where for $k_q < 1$

$$C(k_q, x) = c_1^{2q}\left[\mathcal{B}_{2q}\left(\{Kx\}\right) + (-1)^q(2q)!\pi^{-2q}k_q^{2q} - \mathcal{B}_{2q}\left(0\right)M^{-2q}\right],$$

while for $k_q \geq 1$

$$C(k_q, x) = c_2^{2q}\left[(-1)^q(2q)! + \mathcal{B}_{2q}\left(\{Kx\}\right)\pi^{2q}k_q^{-2q} - 2\mathcal{B}_{2q}\left(0\right)\pi^{2q}(k_qM)^{-2q}\right]$$

and $\int_{-\infty}^{\infty}\mathcal{K}^2\left\{x/h(k_q), t\right\}dt < C^2/\log(\gamma^{-1})$*, for some* $C \in (0,\infty)$ *and* $\gamma \in (0,1)$*, both depending on* k_q*, explicitly given in the proof of Lemma 4 in the Appendix.*

Remarks

1. From the proof of Theorem 3 follows that under the same assumptions it holds also for non-periodic $f(x)$, if $x \in (0,1)$.

2. Integrating the squared bias and the variance results in an alternative expression for the L_2 risk for periodic spline estimators. Namely,

$$\begin{aligned} R(\widehat{f}, f) &= \frac{h(k_q)^{4q}}{(2q)!^2}\|f^{(2q)}C(k_q)\|^2 + o\left\{h(k_q)^{4q}\right\} \\ &+ \frac{\sigma^2}{Nh(k_q)}\int_0^1\int_{-\infty}^{\infty}\mathcal{K}^2\left\{x/h(k_q), t\right\}dtdx + o\left\{N^{-1}h(k_q)^{-1}\right\}, \end{aligned}$$

where $\|\cdot\|$ denotes the $L_2[0,1]$ norm.

In this section we have obtained local asymptotic results for spline estimators of sufficiently smooth fs on $\mathbb{R}$, assuming equidistant design for knots and observations. This assumption is dominating in the literature on equivalent kernels. However, we can follow Huang and Studden (1993) to generalize the result. In particular, if the design points x_i, $i = 1, \ldots, N$ have a limiting density $g(x)$ and the sequence of knots $\underline{\tau}_K$ satisfies $\int_{\tau_{i-1}}^{\tau_i} p(t)dt = 1/K$, for a positive continuous density $p(t)$ on $[0,1]$, then the equivalent kernel for general spline estimator satisfies

$$\mathcal{W}(x,t) = \frac{1}{g(t)h(k_q)/p(t)} \mathcal{K}\left(\frac{x}{h(k_q)/p(t)}, \frac{t}{h(k_q)/p(t)}\right).$$

Another open question so far is the equivalent kernel $\mathcal{K}(x,t)$ on a bounded interval $[0,1]$. However, Theorem 2 emboldens to make some conjectures on this issue. Let $\mathcal{K}_{rs}^{[0,1]}$ and $\mathcal{K}_{ss}^{[0,1]}$ be regression and smoothing splines equivalent kernels on $[0,1]$. For cubic splines Huang and Studden (1993) have shown that $\mathcal{K}_{rs}^{[0,1]} = \mathcal{K}_{rs}(x,t)+\mathcal{K}_{rs}^{b}(x,t)$, where $\mathcal{K}_{rs}^{b}(x,t)$ is some boundary term, not available explicitly. Moreover, $\mathcal{K}_{rs}^{[0,1]}$ satisfies at the boundaries all conditions for boundary kernels as given in Gasser and Müller (1984), confirming that regression spline estimators do not have boundary effects. Messer and Goldstein (1993) obtained $\mathcal{K}_{ss}^{[0,1]} = \mathcal{K}_{ss}(x,t)+\mathcal{K}_{ss}^{b}(x,t)$, where the boundary term $\mathcal{K}_{ss}^{b}(x,t)$ has a complicated closed form expression for each q. This additional term $\mathcal{K}_{ss}^{b}(x,t)$ arises from matching the natural boundary conditions at the end of the interval, illustrating the boundary effects of smoothing spline estimators, unless the regression function satisfies the natural boundary conditions. Since $\mathcal{K}(x,t)$ varies smoothly between $\mathcal{K}_{rs}(x,t)$ and $\mathcal{K}_{ss}(x,t)$, one can expect that additional boundary terms in $\mathcal{K}^{[0,1]}$ also vary smoothly between $\mathcal{K}_{rs}^{b}(x,t)$ and $\mathcal{K}_{ss}^{b}(x,t)$, so that the boundary effects of spline estimators grow as $k_q \to \infty$.

7 Conclusion

We have developed a unified framework that enables to study all (periodic) spline based estimators. Exact expressions for the Demmler-Reinsch basis and corresponding eigenvalues allowed not only to obtain the L_2 risk of periodic spline estimators, but also exact equivalent kernels for all spline estimators on $\mathbb{R}$. Extension of these results to a non-periodic case in the spirit of Rice and Rosenblatt (1983), as well as finding the equivalent kernels on a bounded interval are interesting directions for further research.

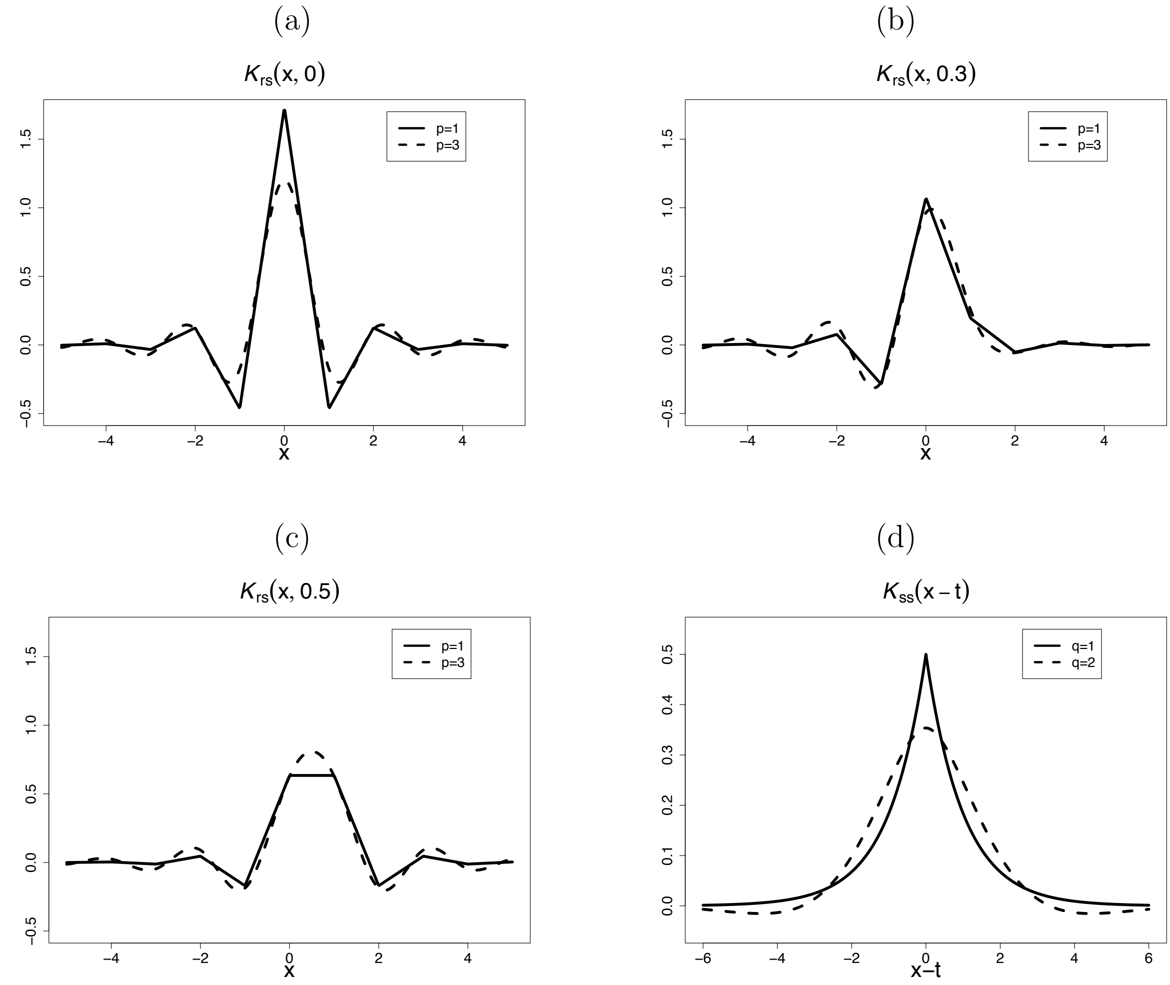

Figure 2: (a) $\mathcal{K}_{rs}(x, 0)$ for $p = 1, 3$, (b) $\mathcal{K}_{rs}(x, 0.3)$ for $p = 1, 3$, (c) $\mathcal{K}_{rs}(x, 0.5)$ for $p = 1, 3$, (d) $\mathcal{K}_{ss}(|x - t|)$ for $q = 1, 2$.

A Proofs

Proof of equation (4)

It is known that $\sum_{l=-\infty}^{\infty}(z+l)^{-1} = \pi\cot(\pi z)^{-1}$ and $\sum_{l=-\infty}^{\infty}(-1)^l(z+l)^{-1} = \pi\sin(\pi z)^{-1}$.

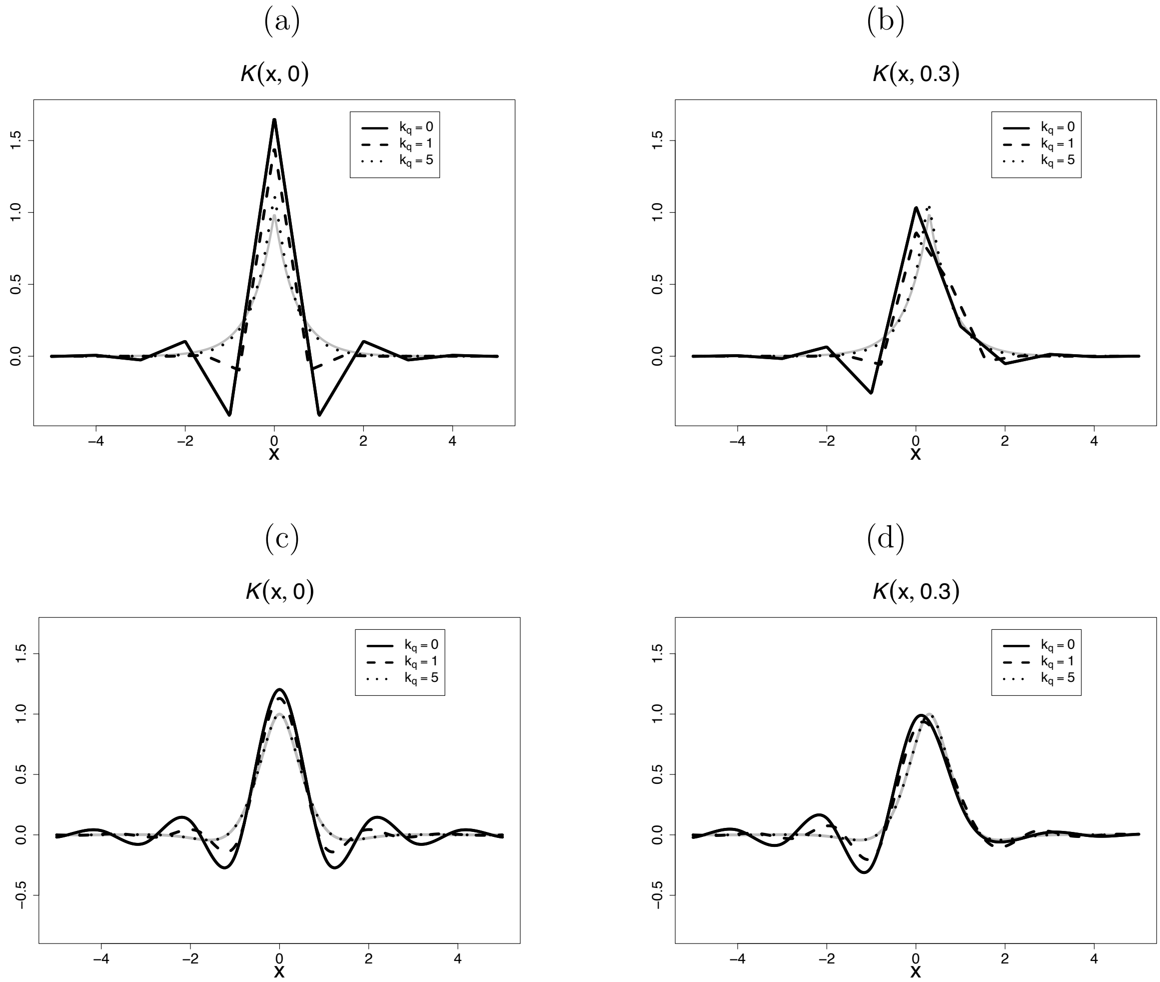

Figure 3: For $k_q = 0, 1, 5$ (a) $\mathcal{K}(x, 0)$ for $M = 5$ and $p = q = 1$ (top) and $p = 2q - 1 = 3$ (bottom), (b) $\mathcal{K}(x, 0.3)$ for $M = 5$ and $p = q = 1$ (top) and $p = 2q - 1 = 3$ (bottom). The grey line corresponds to the smoothing spline kernel.

If p is odd, then

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \text{sinc}\{\pi(z+l)\}^{p+1} &= \frac{\sin(\pi z)^{p+1}}{\pi^{p+1}} \sum_{l=-\infty}^{\infty} (z+l)^{-(p+1)} \\ &= (-1)^p \frac{\sin(\pi z)^{p+1}}{p!\pi^{p+1}} \frac{\partial^p}{\partial z^p} \sum_{l=-\infty}^{\infty} (z+l)^{-1} = (-1)^p \frac{\sin(\pi z)^{p+1}}{p!\pi^p} \frac{\partial^p}{\partial z^p} \cot(\pi z)^{-1}. \end{aligned}$$

If p is even, then

$$\begin{aligned}\sum_{l=-\infty}^{\infty} \operatorname{sinc}\{\pi(z+l)\}^{p+1} &= \frac{\sin(\pi z)^{p+1}}{\pi^{p+1}} \sum_{l=-\infty}^{\infty} (-1)^l (z+l)^{-(p+1)} \\ &= (-1)^p \frac{\sin(\pi z)^{p+1}}{p!\pi^p} \frac{\partial^p}{\partial z^p} \sin(\pi z)^{-1}.\end{aligned}$$

With this one can easily recover the recursive equation for $Q_{p-1}(z)$ for p odd and even.

□

First Q_{p-1} for odd p are given by

$$\begin{aligned}Q_2(z) &= 1/3 + 2\cos(\pi z)^2/3 \\ Q_4(z) &= 2/15 + 11\cos(\pi z)^2/15 + 2\cos(\pi z)^4/15 \\ Q_6(z) &= 17/315 + 4\cos(\pi z)^2/7 + 38\cos(\pi z)^4/105 + 4\cos(\pi z)^6/315,\end{aligned}$$

while for even p

$$\begin{aligned}Q_1(z) &= 1/2 + \cos(\pi z)^2/2 \\ Q_3(z) &= 5/24 + 3\cos(\pi z)^2/4 + \cos(\pi z)^4/24 \\ Q_5(z) &= 61/720 + 479\cos(\pi z)^2/720 + 179\cos(\pi z)^4/720 + \cos(\pi z)^6/720.\end{aligned}$$

Proof of (7) and connection of Q and Euler-Frobenius polynomials

The proof largely follows from Theorem 5 in Lecture 3 of Schoenberg (1973). From formulas (1.1) and (1.4) given in the lecture follows the equality

$$\frac{\exp(2\pi\mathtt{i}z) - 1}{\exp(2\pi\mathtt{i}z) - \exp(x)} \frac{\exp(\{t\}x)}{x^{p+1}} = \sum_{l=0}^{\infty} \frac{\{1 - \exp(-2\pi\mathtt{i}z)\}^{-l} \Phi_l \{t, \exp(2\pi\mathtt{i}z)\}}{\exp(2\pi\mathtt{i}z\lfloor t \rfloor)} x^{l-p-1}.$$

The residue of this function at 0 is

$$\exp\left(-2\pi\mathtt{i}z\left\lfloor t\right\rfloor\right)\left\{1-\exp\left(-2\pi\mathtt{i}z\right)\right\}^{-p}\Phi_p\left\{t,\exp\left(2\pi\mathtt{i}z\right)\right\},$$

while the residues at poles $2\pi\mathtt{i}\left(z+l\right),\ l\in\mathbb{Z}$ are

$$\left\{\exp\left(-2\pi\mathtt{i}z\right)-1\right\}\exp\left\{2\pi\mathtt{i}\{t\}\left(z+l\right)\right\}/\left\{2\pi\mathtt{i}\left(z+l\right)\right\}^{p+1}.$$

With this, from the Cauchy residue theorem, it follows

$$\begin{aligned}&\sum_{l=-\infty}^{\infty}\frac{\left\{1-\exp\left(-2\pi\mathtt{i}z\right)\right\}\exp\left\{2\pi\mathtt{i}\{t\}\left(z+l\right)\right\}}{\left\{2\pi\mathtt{i}\left(z+l\right)\right\}^{p+1}}\\ =&\ \frac{\left\{1-\exp\left(-2\pi\mathtt{i}z\right)\right\}^{-p}\Phi_p\left\{t,\exp\left(2\pi\mathtt{i}z\right)\right\}}{\exp\left(2\pi\mathtt{i}z\left\lfloor t\right\rfloor\right)}.\end{aligned}$$

Multiplication by $\sin\left(\pi z\right)^{p+1}$ and simple simplifications prove (7).

To see the connection between Q and Euler-Frobenius polynomials for odd p, note that after plugging $t=0$ in (6) and (7)

$$\begin{aligned}&p!\exp\left\{\pi\mathtt{i}z\left(p-1\right)\right\}\Pi_p\left\{\exp\left(-2\pi\mathtt{i}z\right)\right\}\\ =&\sum_{l=-\infty}^{\infty}\left(-1\right)^{l(p+1)}\operatorname{sinc}\left\{\pi\left(z+l\right)\right\}^{p+1}.\end{aligned}\tag{25}$$

For even p use (25) and the fact that

$$\begin{aligned}\sum_{l=-\infty}^{\infty}\operatorname{sinc}\left\{\pi\left(z+l\right)\right\}^{p+1}=&\cos\left(\pi z/2\right)^{p+1}\sum_{l=-\infty}^{\infty}\left(-1\right)^{l}\operatorname{sinc}\left\{\pi\left(z/2+l\right)\right\}^{p+1}\\&-\sin\left(\pi z/2\right)^{p+1}\sum_{l=-\infty}^{\infty}\left(-1\right)^{l}\operatorname{sinc}\left\{\pi\left(\left(z+1\right)/2+l\right)\right\}^{p+1}.\end{aligned}$$

□

Proof of Lemma 1

Plugging the Fourier series of a periodic B-spline (3) into the discrete Fourier transform

(DFT) of B-splines, we find

$$
\begin{aligned}
\sum_{i=1}^{K} B_i(x)\exp(-2\pi\mathtt{i}li/K) &= \sum_{m=-\infty}^{\infty} \exp(-2\pi\mathtt{i}mx)\mathrm{sinc}(\pi m/K)^{p+1} \\
&\times \sum_{i=1}^{K} \exp\{2\pi\mathtt{i}i(m-l)/K\} \\
&= K\sum_{n=-\infty}^{\infty} \exp\{-2\pi\mathtt{i}(l+nK)x\}\mathrm{sinc}\{\pi(l/K+n)\}^{p+1} \\
&= K\sqrt{Q_{p,M}(l/K)}\phi_l(x),
\end{aligned}
$$

where in the last equality the representation (13) has been used and $n = (m-l)/K$. The properties of DFT ensure that the functions $\phi_i(x)$, $i = 1,\ldots,K$ are also the basis in $S_{\mathtt{per}}(p;\underline{\tau}_K)$. Property (10) follows immediately from the definition of $Q_{p,M}(z)$. To show the property (11) one can use again the representation in (13) to find

$$
\begin{aligned}
\sqrt{Q_{p,M}(i/K)}\phi_i^{(q)}(x) &= (-2\pi\mathtt{i}i)^q\mathrm{sinc}(\pi i/K)^q \\
&\times \sum_{l=-\infty}^{\infty} (-1)^{lq}\mathrm{sinc}\{\pi(i/K+l)\}^{p+1-q}\exp\{-2\pi\mathtt{i}x(i+lK)\},
\end{aligned}
$$

which implies the assertion and proves the lemma. □

Proof of Theorem 1

From the Parseval's identity

$$
\begin{aligned}
\mathrm{Var}\{\widehat{f}(x)\} &= \sum_{i=1}^{K}\sum_{l=-\infty}^{\infty}\mathrm{Var}(c_{i+lK}) = \sum_{i=1}^{K}\sum_{l=-\infty}^{\infty}\frac{\mathrm{sinc}\{\pi(l+i/K)\}^{2p+2}}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2}\mathrm{Var}(\widehat{y}_i) \\
&= \frac{\sigma^2}{N}\sum_{i=1}^{K}\frac{Q_{2p}(i/K)}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2}.
\end{aligned}
$$

Similar to developments in Section 3, the Fourier coefficients of $s_{\mathtt{p}}(x)$ can be written as $\mathrm{sinc}\{\pi(l+i/K)\}^{p+1}\widetilde{f}_i/\sqrt{Q_{p,M}(i/K)}$ for the Fourier coefficients $\widetilde{f}_i = Q_{p,M}^{1/2}(i/K)Q_{2p}^{-1/2}(i/K)\int_0^1 f(x)\phi_i$

With this,

$$
\begin{aligned}
& \int_0^1 [\mathrm{E}\{\widehat{f}(x)\} - s_{\mathtt{p}}(x)]^2 dx \\
= & \sum_{i=1}^{K} \sum_{l=-\infty}^{\infty} \left| E(c_{i+l/K}) - \mathrm{sinc}\{\pi(l+i/K)\}^{p+1} \widetilde{f}_i / \sqrt{Q_{p,M}(i/K)} \right|^2 \\
= & \sum_{i=1}^{K} \sum_{l=-\infty}^{\infty} \frac{\mathrm{sinc}\{\pi(l+i/K)\}^{2p+2}}{Q_{p,M}(i/K)} \left| \frac{\widehat{f}_i}{1+\lambda\mu_i} - \widetilde{f}_i \right|^2 \\
= & \sum_{i=1}^{K} \frac{Q_{2p}(i/K)(\lambda\mu_i)^2 |\widetilde{f}_i|^2}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2} \left| 1 - \frac{\widehat{f}_i/\widetilde{f}_i - 1}{\lambda\mu_i} \right|^2,
\end{aligned}
$$

with $\widehat{f}_i = Q_{p,M}^{1/2}(i/K) Q_{2p}^{-1/2}(i/K) N^{-1} \sum_{l=1}^{N} f(l/N)\phi_i(l/N)$, proving the theorem. □

Proof of Corollary 1

Let $\lambda\mu_{K/2} = O(1)$. Then, since $Q_{2p}(i/K)/Q_{p,M}(i/K) \leq 1$ for all i,

$$
\mathrm{Var}\{\widehat{f}(x)\} = 2\frac{\sigma^2}{N} \sum_{i=1}^{K/2} \frac{Q_{2p}(i/K)}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2} = O(KN^{-1}).
$$

In case $\lambda\mu_{K/2} \to \infty$ the integrated variance can be bounded by

$$
\begin{aligned}
\mathrm{Var}\{\widehat{f}(x)\} &= 2\frac{\sigma^2}{N} \sum_{i=1}^{K/2} \frac{Q_{2p}(i/K)}{Q_{p,M}(i/K)(1+\lambda\mu_i)^2} \\
&\leq 2\frac{\sigma^2}{N} \sum_{i=1}^{K/2} \frac{1}{\{1+\lambda(4i)^{2q} Q_{2p-2q}(1/2)/Q_{p,M}(1/2)\}^2}.
\end{aligned}
$$

Approximating the latter sum by an integral as in Wahba (1975) will result in the rate $O(\lambda^{-1/(2q)} N^{-1})$ if $\lambda^{1/(2q)} N \to \infty$.

The integrated shrinkage bias in both asymptotic scenarios can be bounded by

$$
\int_0^1 [\mathrm{E}\{\widehat{f}(x)\} - s_{\mathtt{p}}(x)]^2 dx \leq 2\lambda^2 \sum_{i=1}^{K/2} \mu_i^2 |\widetilde{f}_i|^2 = O(\lambda^2).
$$

Optimizing $\mathrm{R}(\widehat{f}, f)$ with respect to the parameters K and λ gives the optimal rates for K and λ in both asymptotic scenarios. $\square$

Proof of Lemma 2

First we aim to represent $W(x,t)$ and $\mathcal{W}(x,t)$ as a ratio of two polynomials of exponential functions. Basis functions $\phi_i(x)$ and $\phi(u,x)$, as well as Q polynomials, can be expressed in terms of the Euler-Frobenius polynomials of exponential functions, as shown in Section 2. Moreover, from (6) and (8) we find that $Q_{p,M}(u) = \exp(2\pi \mathtt{i} pu)\widetilde{\Pi}_{p,M}\{\exp(-2\pi\mathtt{i}u)\}$, where $\widetilde{\Pi}_{p,M}(1) = 1$ and for $u \neq 1$, p even and M odd

$$\begin{aligned} \widetilde{\Pi}_{p,M}(u) &= \sum_{j=0}^{p}\sum_{l=0}^{p} \frac{\Pi_j(u)\Pi_l(u^{-1})u^l(u-1)^{2p-l-j}}{(-1)^{p-l}j!l!\,(p-j)!(p-l)!} \\ &\times \sum_{s=0}^{2p-l-j} \binom{2p-l-j}{s} 2^s \frac{\mathcal{B}_{s+1}\,(M) - \mathcal{B}_{s+1}\,(0)}{(s+1)(2M)^{2p-j-l+1}}, \end{aligned}$$

with $\mathcal{B}_{p+1}$ denoting the $(p+1)$th degree Bernoulli polynomial. In all other cases

$$\widetilde{\Pi}_{p,M}(u) = \sum_{j=0}^{p}\sum_{l=0}^{p} \frac{\Pi_j(u)\Pi_l\,(u^{-1})\,u^l(u-1)^{2p-l-j}}{(-1)^{p-l}j!l!\,(p-j)!\,(p-l)!} \frac{\mathcal{B}_{2p-j-l+1}\,(M) - \mathcal{B}_{2p-j-l+1}\,(0)}{M^{2p-j-l+1}\,(2p-j-l+1)}.$$

For $M > 1$ one can also use $Q_{p,M}(z) = Q_{2p}(z) + O(M^{-p-1})$ and replace $Q_{p,M}(z)$ by a much simpler $Q_{2p}(z)$. With this

$$\begin{aligned} W(x,t) &= \sum_{i=1}^{K} \frac{\exp\left(-2\pi\mathtt{i}d_{x,t}i/K\right)\sum_{l=0}^{2p}\alpha_l(\{Kx\},\{Kt\})\exp\left(-2\pi\mathtt{i}il/K\right)}{P_{2p}\left\{\exp\left(-2\pi\mathtt{i}i/K\right)\right\}} \\ \mathcal{W}(x,t) &= \int_0^K \frac{\exp\left(-2\pi\mathtt{i}d_{x,t}u/K\right)\sum_{l=0}^{2p}\alpha_l(\{Kx\},\{Kt\})\exp\left(-2\pi\mathtt{i}ul/K\right)}{P_{2p}\left\{\exp\left(-2\pi\mathtt{i}u/K\right)\right\}}du. \end{aligned}$$

The coefficients of the partial fractional decomposition of $1/P_{2p}$ are $1/P_{2p}^{'}\,(r_j)$ and $1/P_{2p}^{'}\left(r_j^{-1}\right)$ correspondent to the roots r_j and r_j^{-1} for $j = 1,\dots,p$. From the representation of P_{2p} as a function of $\cos^2(\pi i/K) = \{\exp\left(-2\pi\mathtt{i}i/K\right) + \exp\left(2\pi\mathtt{i}i/K\right) + 2\}/4$ follows that

$P'_{2p}(r_i^{-1}) = -r_i^{2-2p} P'_{2p}(r_i^{-1})$. Then

$$W(x,t) = \sum_{j=1}^{p} \sum_{l=0}^{2p} \frac{\alpha_l\left(\{Kx\},\{Kt\}\right)}{P'_{2p}\left(r_j\right)} R_n(j,l)$$
$$\mathcal{W}(x,t) = \sum_{j=1}^{p} \sum_{l=0}^{2p} \frac{\alpha_l(\{Kx\},\{Kt\})}{P'_{2p}\left(r_j\right)} R(j,l),$$

for

$$\begin{aligned} R_n(j,l) &= \sum_{i=1}^{K} \left[\frac{\exp\left\{-2\pi\mathtt{i}\left(d_{x,t}+l\right)i/K\right\}}{\exp(-2\pi\mathtt{i}i/K) - r_j} - \frac{r_j^{2p-2}\exp\left\{-2\pi\mathtt{i}\left(d_{x,t}+l\right)i/K\right\}}{\exp(-2\pi\mathtt{i}i/K) - r_j^{-1}} \right] \\ R(j,l) &= \int_0^K \left[\frac{\exp\left\{-2\pi\mathtt{i}\left(d_{x,t}+l\right)u/K\right\}}{\exp(-2\pi\mathtt{i}u/K) - r_j} - \frac{r_j^{2p-2}\exp\left\{-2\pi\mathtt{i}\left(d_{x,t}+l\right)u/K\right\}}{\exp(-2\pi\mathtt{i}u/K) - r_j^{-1}} \right] du. \end{aligned}$$

To find $R_n(j,l)$ we use the inverse discrete Fourier transform of the sequence r_j^i, $i = 1, \dots K$, which can be obtained from the geometric progression formula, so that

$$R_n(j,l) = K\, \frac{r_j^{(d_{x,t}+l-1) \bmod K} + r_j^{K+2p-2-(d_{x,t}+l-1) \bmod K}}{1 - r_j^K}.$$

$R(j,l)$ follows from the Cauchy integral formula, where the contour integral is taken counter-clockwise

$$R(j,l) = \begin{cases} \frac{K}{(2\pi\mathtt{i})} \oint_{|z|=1} \left[\frac{z^{d_{x,t}+l-1}}{z-r_j} - \frac{r_j^{2p-2} z^{d_{x,t}+l-1}}{z-r_j^{-1}} \right] du = K r_j^{d_{x,t}+l-1}, & (d_{x,t}+l) > 0 \\ \frac{K}{(2\pi\mathtt{i})} \oint_{|z|=1} \left[\frac{r_j^{2p-1} z^{-(d_{x,t}+l)}}{z-r_j} - \frac{r_j^{-1} z^{-d_{x,t}-l}}{z-r_j^{-1}} \right] du = K z^{-d_{x,t}-l+2p-1} & (d_{x,t}+l) < 0. \end{cases}$$

□

Proof of Lemma 3

From (21) and symmetry of the kernel $\mathcal{W}(x,t) = \mathcal{W}(t,x)$ follows that

$$\mathcal{W}(x,t) = \int_{-\infty}^{\infty} \overline{a(u,x)} \exp\left(-2\pi i t u\right) du,$$

with $a(u,x)$ defined as

$$\begin{aligned} a\left(u,x\right) &= \frac{\operatorname{sinc}\left\{\pi\left(u/K\right)\right\}^{p+1}\phi\left(u,x\right)}{\sqrt{Q_{p,M}\left(u/K\right)}\left\{1+\lambda\mu\left(u\right)\right\}} \\ &= \int_0^1 W\left(x,t\right)\exp\left(-2\pi\mathtt{i}tu\right)dt,\ \ u\in\mathbb{R}, \end{aligned} \tag{26}$$

where the last equality is obtained using (18) and the Poisson summation formula. Note that for $u \in \mathbb{Z}$, function $a\left(u,x\right)$ coincides with the u-th Fourier coefficient of $W\left(x,t\right)$ given in (19). Properties of the Fourier transform and (26) ensure that

$$\begin{aligned} \int_{-\infty}^{\infty}(2\pi\mathtt{i}t)^m\mathcal{W}(x,t)\exp(2\pi\mathtt{i}tu)dt &= \int_0^1(2\pi\mathtt{i}t)^m W(x,t)\exp(2\pi\mathtt{i}tu)dt \\ &= \frac{\partial^m}{\partial u^m}\left\{\overline{a(u,x)}\right\}. \end{aligned}$$

Evaluating derivative of $a\left(u,x\right)$ at $u=0$ and grouping the terms we represent

$$\int_{-\infty}^{\infty}(2\pi\mathtt{i}t)^m\mathcal{W}(x,t)dt = I_1+I_2+I_3, \tag{27}$$

where

$$\begin{aligned} I_1 =& \frac{\partial^m}{\partial u^m}\left[\exp\left(2\pi\mathtt{i}xu\right)\left\{1+\frac{\operatorname{sinc}\left(\pi u/K\right)^{2p+2}-Q_{p,M}\left(u/K\right)}{Q_{p,M}\left(u/K\right)}\right\}\right]_{u=0} \\ I_2 =& \frac{\partial^m}{\partial u^m}\left[\frac{\left\{\sin\left(\pi u/K\right)\operatorname{sinc}\left(\pi u/K\right)\right\}^{p+1}}{Q_{p,M}\left(u/K\right)}\sum_{l\neq 0}\frac{\exp\left\{2\pi\mathtt{i}x\left(u+lK\right)\right\}}{\left\{\left(-1\right)^l\pi\left(u/K+l\right)\right\}^{p+1}}\right]_{u=0} \\ I_3 =& \frac{\partial^m}{\partial u^m}\left[-\frac{\lambda\mu\left(u\right)\operatorname{sinc}\left(\pi u/K\right)^{p+1}\phi\left(u,x\right)}{\sqrt{Q_{p,M}\left(u/K\right)}\left\{1+\lambda\mu\left(u\right)\right\}}\right]_{u=0}. \end{aligned}$$

The idea is to represent each of these components as a product of $\sin\left(\pi u/K\right)^n$, $n\in\mathbb{Z}$ and some function that is differentiable at 0. Here we use that $Q_{p,M}(0)=\mu(0)=\phi(0,x)=1$,

$$\left.\frac{\partial^m}{\partial u^m}\sin\left(\pi u/K\right)^n\right|_{u=0} = \begin{cases} 0, & m=0,\dots n-1 \\ n!\left(\pi/K\right)^n, & m=n \end{cases}$$

and the Fourier series of the periodic Bernoulli polynomials $\mathcal{B}_{p+1}(\{x\}) = (-1)^p(p+1)!\sum_{s\neq 0}\exp(-2\pi\mathtt{i}sx)/\left(2\pi\mathtt{i}s\right)^{p+1}$. Moreover, to handle I_1, the following expression for $Q_{p,M}$ is needed

$$Q_{p,M}\left(z\right)=\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\operatorname{sinc}\left\{\pi\left(z+l\right)\right\}^{p+1}\operatorname{sinc}\left\{\pi\left(z+l+jM\right)\right\}^{p+1},$$

what can be obtained using the series of the exponential splines (7). After regrouping, the last expression becomes

$$\begin{aligned}Q_{p,M}\left(u/K\right) &= \operatorname{sinc}\left(\pi u/K\right)^{2p+2}+2\left\{\sin\left(\pi u/K\right)\operatorname{sinc}\left(\pi u/K\right)\right\}^{p+1}\\ &\times \sum_{l\neq 0}\frac{\left(-1\right)^{lM(p+1)}}{\left\{\pi\left(u/K+lM\right)\right\}^{p+1}}+\frac{\sin\left(\pi u/K\right)^{2p+2}}{\pi^{2p+2}}\\ &\times \left[\sum_{j=1}^{M-1}\left\{\sum_{l=-\infty}^{\infty}\frac{\left(-1\right)^{(j+lM)(p+1)}}{\left(u/K+j+lM\right)^{p+1}}\right\}^2+\left\{\sum_{l\neq 0}\frac{\left(-1\right)^{lM(p+1)}}{\left(u/K+lM\right)^{p+1}}\right\}^2\right].\end{aligned}$$

Putting it all together,

$$I_1=\begin{cases}\left(2\pi\mathtt{i}x\right)^m, & m=0,\dots p\\ \left(2\pi\mathtt{i}x\right)^{p+1}+2(2\pi\mathtt{i}/N)^{p+1}\mathcal{B}_{p+1}\left(\left\{\frac{p+1}{2}\right\}\right), & m=p+1.\end{cases}$$

The expression for I_2 follows immediately from its representation

$$I_2=\begin{cases}0, & m=0,\dots p\\ -(2\pi\mathtt{i}/K)^{p+1}\mathcal{B}_{p+1}\left(\left\{Kx+\frac{p+1}{2}\right\}\right), & m=p+1.\end{cases}$$

To find I_3, we use $\mu\left(u\right)=(2K)^{2q}\sin(\pi u/K)^{2q}Q_{2p-2q}(u/K)/Q_{p,M}(u/K)$

$$I_3=\begin{cases}0, & m=0,\dots 2q-1\\ -\lambda\left(2\pi\right)^{2q}\left(2q\right)!, & m=2q.\end{cases}$$

To get the result for $\int_{-\infty}^{\infty}\left(t-x\right)^m\mathcal{W}\left(x,t\right)dt=\int_0^1\left(t-x\right)^m W\left(x,t\right)dt$ one needs to expand $(t-x)^m$ and use (27). $\square$

Proof of Theorem 2

$$\mathcal{W}(x,t) = \int_0^K \frac{\phi(u,x)\overline{\phi(u,t)}}{1+\lambda\mu(u)}du = \Re\int_0^{1/2}\frac{2K\phi(Ku,x)\overline{\phi(Ku,t)}}{1+\lambda\mu(Ku)}du.$$

Let first consider $0 \le k_q < 1$. Scaling $\mathcal{W}(x,t)$ with $c_1^{-1}K^{-1}$, leads to

$$\begin{aligned} c_1\mathcal{K}(c_1x,c_1t) &= \Re\left[\int_0^{1/2} 2\phi(Ku,x/K)\overline{\phi(Ku,t/K)}du\right. \\ &- \left.\int_0^{1/2}\frac{2\lambda\mu(Ku)\phi(Ku,x/K)\overline{\phi(Ku,t/K)}}{1+\lambda\mu(Ku)}du\right] \\ &= \mathcal{K}_{rs}(x,t) - k_q^{2q}\mathcal{K}_1(x,t), \end{aligned}$$

where $\mathcal{K}_{rs}(x,t)$ is the equivalent regression spline kernel on $\mathbb{R}$ and

$$\begin{aligned} \mathcal{K}_1(x,t) &= \Re\int_0^{1/2}\frac{2\sin(\pi u)^{2q}Q_{2q-2}(u)\phi(Ku,x/K)\overline{\phi(Ku,t/K)}}{\pi^{2q}Q_{p,M}(u)\{1+\lambda\mu(Ku)\}}du \\ &\le \frac{2^{2q}Q_{2q-2}(1/2)}{\pi^{2q}Q_{4q-2}(1/2)}\mathcal{K}_{rs}(x,t). \end{aligned}$$

Using $Q_{lq-2}(1/2) = 2\pi^{lq}(2^{lq}-1)\zeta(lq)$ for the Riemann zeta function $\zeta(lq) = \sum_{i=1}^{\infty} i^{-lq}$, one can get explicit bounds for each q. For $k_q \ge 1$ we first introduce the following notation.

1. $1+\lambda\mu(Ku) = \{1+\lambda(2\pi Ku)^{2q}\}\{1+r_1(u)\}$
2. $\phi(Ku,x)\overline{\phi(Ku,t)} = \exp\{2\pi\mathtt{i}Ku(x-t)\}\{1+r_2(x,t,u)\}$
3. $r_q(x,t,u) = \{r_2(x,t,u)-r_1(u)\}\{1+r_1(u)\}^{-1}$

Scaling $\mathcal{W}(x,t)$ with $c_2^{-1}\lambda^{1/(2q)}$ and using approximations defined above results in

$$\begin{aligned} c_2\mathcal{K}(c_2x,c_2t) &= \int_{-\infty}^{\infty}\frac{\exp\{2\pi\mathtt{i}u(t-x)\}}{1+(2\pi u)^{2q}}du \\ &+ \Re\int_0^{k_q/2}\frac{2\exp\{2\mathtt{i}u(t-x)\}}{\pi\{1+(2u)^{2q}\}}r_q(x,t,u/k_q)du \\ &- \Re\int_{k_q/2}^{\infty}\frac{2\exp\{2\mathtt{i}u(t-x)\}}{\pi\{1+(2u)^{2q}\}}du = \mathcal{K}_{ss}(x,t) + k_q^{-2q+1}\mathcal{K}_2(x,t), \end{aligned}$$

where $\mathcal{K}_{ss}(x,t)$ is the smoothing spline kernel on $\mathbb{R}$ and

$$\begin{aligned}\pi\mathcal{K}_2(x,t) &= k_q^{2q-1}\Re\int_0^{k_q/2}\frac{2\exp\{2\mathrm{i}u(t-x)\}}{1+(2u)^{2q}}r_q(x,t,u/k_q)du \\ &- \int_1^\infty \frac{\cos\{k_q u(t-x)\}}{k_q^{-2q}+u^{2q}}du.\end{aligned}$$

The second component of $\pi\mathcal{K}_2(x,t)$ is obviously bounded by 1. Now, let us consider $r_q(x,t,u/k_q)$. Since $Q_{4q-2}(u) \leq Q_{p,M}(u) \leq Q^2_{2q-2}(u)$ for odd p, the term $r_1(u)$ is bounded by

$$\frac{(2uk_q)^{2q}}{1+(2uk_q)^{2q}}\left\{\frac{\mathrm{sinc}(\pi u)^{2q}}{Q_{2q-2}(u)}-1\right\} \leq r_1(u) \leq \frac{(2uk_q)^{2q}}{1+(2uk_q)^{2q}}\left\{\frac{\mathrm{sinc}(\pi u)^{2q}Q_{2q-2}(u)}{Q_{4q-2}(u)}-1\right\},$$

so that $|r_1(u)| \leq 1-\mathrm{sinc}(\pi u)^{2q}/Q_{2q-2}(u)$. Observing

$$Q_{2q-2}(u/k_q)\mathrm{sinc}(\pi u/k_q)^{-2q} = 1+2\zeta(2q)(u/k_q)^{2q}+O(k_q^{-4q}),$$

we obtain $|r_1(u/k_q)| \leq \zeta(2q)(2k_q)^{-2q}+O(k_q^{-4q})$. Using the same techniques gives $|r_2(x,t,u/k_q)| \leq 8\zeta(2q)(2k_q)^{-2q}+O(k_q^{-4q})$. In principle, one can also find a lower bound for $1+r_1(u/k_q)$ depending on q and k_q, but it is enough to note that $1+r_1(u/k_q) \geq 1/2$. Finally, $|r_q(x,t,u/k_q)| \leq 18\zeta(2q)(2k_q)^{-2q}+O(k_q^{-4q})$ and hence, the first term in $\pi\mathcal{K}_2(x,t)$ is also bounded for any $k_q \geq 1$.

It remains to set $\mathcal{K}_{ss}(x,t)$ and $\mathcal{K}_{rs}(x,t)$. Thomas-Agnan (1996) already treated asymptotic equivalent kernels for smoothing splines on $\mathbb{R}$ and obtained the formula for $\mathcal{K}_{ss}(x,t)$ given in this theorem. $\mathcal{K}_{rs}(x,t)$ is obtained from (23), scaling $\mathcal{W}(x,t)$ with K and setting $P_{2p}(u) = \Pi_{2p}(u)$. □

The following lemma will be used in the proof of Theorem 3.

Lemma 4 *Kernel $\mathcal{K}(x,t)$, $x,t \in \mathbb{R}$ decays exponentially, i.e. there are constants* $0 <$

$C < \infty$ *and* $0 < \gamma < 1$ *such that*

$$|\mathcal{K}(x,t)| < C\gamma^{|x-t|}.$$

Proof of Lemma 4

Since $\mathcal{K}(x,t)$ is defined as a scaled with $h(k_q)$ function $\mathcal{W}(x,t)$, from (23) and (24) one finds for $k_q < 1$ that

$$c_1\mathcal{K}(c_1x, c_1t) = \sum_{j=1}^{p}\sum_{l=0}^{2p} \frac{\alpha_l\left(\{x\}, \{t\}\right)}{P'_{2p}(r_j)} r_j^{|\lfloor x\rfloor - \lfloor t\rfloor + l - 1| + I_{\{\lfloor x\rfloor - \lfloor t\rfloor \le -l\}}(2p-2)},$$

while for $k_q \geq 1$,

$$\pi c_2\mathcal{K}(\pi c_2 x, \pi c_2 t) = k_q \sum_{j=1}^{p}\sum_{l=0}^{2p} \frac{\alpha_l\left(\{xk_q\}, \{tk_q\}\right)}{P'_{2p}(r_j)} r_j^{|\lfloor xk_q\rfloor - \lfloor tk_q\rfloor + l - 1| + I_{\{\lfloor xk_q\rfloor - \lfloor tk_q\rfloor \le -l\}}(2p-2)}.$$

Here polynomial P_{2p} given in (22), $r_j = r_j(k_q)$ is a root of P_{2p} with $|r_j| < 1$. If k_q is a bounded constant then $r_j = r_j(k_q) \nrightarrow \exp(-2\pi\mathtt{i}u),\ u \in (0,1)$ since

$$P_{2p}\left\{\exp(-2\pi\mathtt{i}u)\right\} = \exp(-2\pi\mathtt{i}pu)\left\{Q_{p,M}(u) + (2k_q/\pi)^{2q}\sin(\pi u)^{2q}Q_{2p-2q}(u)\right\} \neq 0,$$

where the relationship between Euler-Frobenius and Q-polynomials has been used. Similarly, $r_j = r_j(k_q) \nrightarrow 0$ and $0 < \gamma < 1$ can be defined as

$$\gamma = \begin{cases} \sup_{j,k_q} |r_j(k_q)|, & k_q < 1 \\ \sup_{j,k_q} \left|r_j(k_q)^{k_q}\right|, & 1 \leq k_q < \infty, \end{cases}$$

while

$$C = \sup_{k_q,j} \frac{p(2p+1)\sup_{l,x,t}\alpha_l(\{x\},\{t\})}{\left|P'_{2p}\{r_j(k_q)\}\right| |r_j(k_q)|^{l+1}} < \infty.$$

For $k_q \to \infty$ it is known from Theorem 2 that $\lim_{k_q\to\infty}\mathcal{K}(x,t) = \mathcal{K}_{ss}\left\{(x-t)/\widetilde{c}_2\right\}/\widetilde{c}_2$. To obtain the bound on the smoothing spline kernel $\mathcal{K}_{ss}(x)$, the expression given in Theorem

2 can be rewritten as

$$
\begin{aligned}
|\mathcal{K}_{ss}(x-t)| &= \Bigg| -I_{\{q\ is\ odd\}} \frac{\exp\left(-|x-t|\right)}{2q} \\
&+ \sum_{j=0}^{\lfloor (q-1)/2 \rfloor} \frac{\exp\left[-|x-t|\sin\left\{\pi(2j+1)/(2q)\right\}\right]}{q} \\
&\times \sin\left[\frac{\pi(2q-1)(2j+1)}{2q} - |x-t|\cos\left\{\frac{\pi(2j+1)}{2q}\right\}\right]\Bigg| \\
&\leq \frac{q+1}{2q}\exp\left\{-|x-t|\sin\left(\pi/2q\right)\right\},
\end{aligned}
$$

so that one can set $\gamma = \exp\left[-\sin\{\pi/(2q)\}/\widetilde{c}_2\right] \in (0,1)$ and $C = (q+1)/(2q\widetilde{c}_2) < \infty$ for $k_q \to \infty$. $\square$

Proof of Theorem 3

$$
\begin{aligned}
\mathrm{E}\left\{\widehat{f}(x)\right\} &= \mathrm{E}\left\{\frac{1}{N}\sum_{i=1}^{N} W(x, i/N)Y_i\right\} = \int_0^1 W(x,t)f(t)dt + O\left(N^{-1}\right) \\
&= \int_{-\infty}^{\infty} \mathcal{W}\left(x,t\right) f(t)dt + O\left(N^{-1}\right).
\end{aligned}
$$

Expanding $f(t)$ in a Taylor series around x and using Lemma 3 results in

$$
\begin{aligned}
\mathrm{E}\left\{\widehat{f}(x)\right\} - f(x) &= \int_{-\infty}^{\infty} \mathcal{W}\left(x,t\right)\left(x-t\right)^{2q} \frac{f^{(2q)}(\xi_{x,t})}{(2q)!} dt + O(N^{-1}) \\
&= \frac{f^{(2q)}(x)}{(2q)!} \int_{-\infty}^{\infty} \mathcal{W}\left(x,t\right)\left(x-t\right)^{2q} dt + R_\xi(x) + O(N^{-1}) \\
&= h(k_q)^{2q} \frac{f^{(2q)}(x)}{(2q)!} \int_{-\infty}^{\infty} \mathcal{K}\left(x_h, t_h\right)\left(x_h - t_h\right)^{2q} dt_h \\
&+ R_\xi(x) + O(N^{-1}),
\end{aligned}
$$

where $\xi_{x,t}$ is a point between x and t, $\int_{-\infty}^{\infty} \mathcal{K}(x_h, t)(x_h - t)^{2q}\, dt = -C(k_q, x)$ given in the Theorem 3, $x_h = x/h(k_q)$, $t_h = t/h(k_q)$ and

$$R_\xi(x) = h(k_q)^{2q} \int_{-\infty}^{\infty} \mathcal{K}(x_h, t_h)(x_h - t_h)^{2q} \frac{f^{(2q)}(\xi_{x,t}) - f^{(2q)}(x)}{h(k_q)(2q)!} dt.$$

It remains to show that $R_\xi(x) = o\{h(k_q)^{2q}\}$. Using techniques similar to Huang and Studden (1993),

$$\begin{aligned} R_\xi(x) &= h(k_q)^{2q} \sum_{l=-\infty}^{\infty} \int_{x+(l-1)h}^{x+lh} \mathcal{K}(x_h, t_h)(x_h - t_h)^{2q} \frac{f^{(2q)}(\xi_{x,t}) - f^{(2q)}(x)}{h(k_q)(2q)!} dt \\ &\leq h(k_q)^{2q+\alpha} CL \sum_{l=-\infty}^{\infty} \int_{x+(l-1)h}^{x+lh} \gamma^{|x_h - t_h|} \frac{|x_h - t_h|^{2q+\alpha}}{h(k_q)(2q)!} dt \\ &\leq h(k_q)^{2q+\alpha} \frac{2CL}{(2q)!} \sum_{l=1}^{\infty} \gamma^{l-1} l^{2q+\alpha} = o\left\{h(k_q)^{2q}\right\}, \end{aligned}$$

where the exponential bound on the kernel from Lemma 4 together with the Hölder continuity of $f^{(2q)}$ have been used. Next, the variance of $\widehat{f}(x)$ is given by

$$\operatorname{Var}\left\{\widehat{f}(x)\right\} = \frac{\sigma^2}{N^2} \sum_{i=1}^{N} W^2(x, i/N) = \frac{\sigma^2}{N} \int_0^1 W^2(x,t)dt + O(N^{-1}).$$

Let us define $\mathtt{K}(x,t)$ via

$$h(k_q)^{-1}\mathtt{K}(x_h, t_h) = W(x,t) = \sum_{l=-\infty}^{\infty} \mathcal{W}(x, t+l) = h(k_q)^{-1} \sum_{l=-\infty}^{\infty} \mathcal{K}(x_h, t_h + l_h),$$

for $l_h = l/h(k_q)$. Then, using periodicity of $W(x,t)$

$$\begin{aligned} \int_0^1 W^2(x,t)dt &= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} W^2(x,t)dt = \frac{1}{h(k_q)^2} \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathtt{K}^2(x_h, t_h)dt \\ &= \frac{1}{h(k_q)} \left\{ \int_{-\infty}^{\infty} \mathcal{K}^2(x_h, t)dt + R_k(x) \right\}, \end{aligned}$$

for

$$\begin{aligned} h(k_q)R_k(x) &= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathtt{K}^2(x_h,t_h)dt - \int_{-\infty}^{\infty} \mathcal{K}^2(x_h,t_h)dt \\ &= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \left\{\mathtt{K}^2(x_h,t_h) - \mathcal{K}^2(x_h,t_h)\right\} dt \\ &- \int_{-\infty}^{x-\frac{1}{2}} \mathcal{K}^2(x_h,t_h)dt - \int_{x+\frac{1}{2}}^{\infty} \mathcal{K}^2(x_h,t_h)dt. \end{aligned}$$

Now, we can make use of $\mathtt{K}(x,t) = \sum_{l=-\infty}^{\infty} \mathcal{K}(x,t+l)$ and of the exponential decay of $\mathcal{K}(x,t)$ found in Lemma 4 to bound terms in $h(k_q)R_k(x)$. That is,

$$\begin{aligned} &\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \left\{\mathtt{K}^2(x_h,t_h) - \mathcal{K}^2(x_h,t_h)\right\} dt \\ =& \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \sum_{l\neq 0} \mathcal{K}(x_h,t_h+l_h) \left\{\sum_{l\neq 0} \mathcal{K}(x_h,t_h+l_h) + 2\mathcal{K}(x_h,t_h)\right\} dt \\ \leq& \ C^2 \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \sum_{l\neq 0} \gamma^{|x_h-t_h-l_h|} \left(\sum_{l\neq 0} \gamma^{|x_h-t_h-l_h|} + 2\gamma^{|x_h-t_h|}\right) dt \\ \leq& \ h(k_q) \frac{C^2\gamma^{1/h(k_q)}\left\{4 + 2h(k_q)^{-1}\log(\gamma^{-1})\right\}}{\left\{\gamma^{1/h(k_q)} - 1\right\}^2 \log(\gamma^{-1})}, \end{aligned}$$

where the sum under the integral

$$\sum_{l\neq 0} \gamma^{|x_h-t_h-l_h|} = \left(\gamma^{t_h-x_h} + \gamma^{x_h-t_h}\right)\gamma^{1/h(k_q)} / \left\{1 - \gamma^{1/h(k_q)}\right\},$$

for $t \in [x-1/2, x+1/2]$ has been used. Also,

$$\begin{aligned} \int_{-\infty}^{x-\frac{1}{2}} \mathcal{K}^2(x_h,t_h)dt &+ \int_{x+\frac{1}{2}}^{\infty} \mathcal{K}^2(x_h,t_h)dt \\ &\leq C^2 \left\{\int_{-\infty}^{x-\frac{1}{2}} \gamma^{2(x_h-t_h)}dt + \int_{x+\frac{1}{2}}^{\infty} \gamma^{2(t_h-x_h)}dt\right\} \\ &= h(k_q)\frac{C^2\gamma^{1/h(k_q)}}{\log(\gamma^{-1})}. \end{aligned}$$

In a similar fashion one finds $\int_{-\infty}^{\infty} \mathcal{K}^2(x_h, t)dt \leq C^2/\log(\gamma^{-1})$. Putting it all together gives

$$
\begin{aligned}
|R_k(x)| &\leq \frac{C^2\gamma^{1/h(k_q)}}{\log(\gamma^{-1})}\left[1 + \frac{4 + 2h(k_q)^{-1}\log(\gamma^{-1})}{\{\gamma^{1/h(k_q)} - 1\}^2}\right] \\
&= O\left\{h(k_q)^{-1}\gamma^{1/h(k_q)}\right\} = o(1),
\end{aligned}
$$

proving the theorem. □

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