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A new inequality-sensitive multidimensional deprivation index (*MDI*) for dichotomous variables*

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Abstract

In this paper, we propose to use the so-called Sen-Shorrocks poverty index (Shorrocks, 1995) to measure multidimensional deprivation when only dichotomous variables are available to assess deprivation in the various deprivation domains, the most common case in the literature, and introduce a rank-dependent multidimensional poverty index for multiple binary indicators using a counting approach. The resulting multidimensional deprivation index, or *MDI* for short, also has a nice graphical representation that is derived from the *TIP curve* of Jenkins and Lambert (1997). The great advantage of measuring multidimensional deprivation using the *MDI* is that this index is sensitive to inequality and can be fully broken down by deprivation domain, as well as by population subgroups, two features that have far-reaching policy implications and have proven to be important for poverty analysis. An empirical illustration based on deprivation data from four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua) shows the usefulness of the *MDI*, as it allows us to conclude, for example, that in each country, education contributes the most (about 30%) to multidimensional poverty.

Keywords: Multidimensional poverty analysis; Inequality; Gini index; Dominance

JEL Codes: I3; I31; I32; D6; D63; O1

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1 Introduction

Poverty is one of the main sources of unfreedom (Sen, 2000a); it can entail not only the absence of material needs for well-being, but also the denial of opportunities and rights to live a tolerable life (Anand & Sen, 1997). It is, in many ways, "the worst form of human deprivation" (p. 4). The alleviation of poverty, as well as of inequalities in its multiple dimensions, therefore remains the key objective of development policy all over the world; this has been emphasized, for example, by the post-2015 development agenda ("Transforming our world: the 2030 Agenda for Sustainable Development") agreed by the global community on September 25, 2015, which recognizes that the elimination of poverty is the greatest global challenge and an essential prerequisite for sustainable development (UN, 2015, p. 1).

To understand the threat posed by the problem of poverty, it is necessary to know the interdependencies of the dimensions of poverty, its determinants, and the process through which it appears to deepen. In this context, an important question is: how to measure poverty in a society and its changes (Chakravarty, 2006), as policy change is often based on it. Poverty measurement, our central concern in this paper, can be of great importance for the orientation and monitoring of poverty alleviation policies; it is necessary, if not sufficient, for any reasoned evaluation of these policies and can be of "enormous practical relevance" (Alkire & Foster, 2011a, p. 290): what we measure affects what we do, and if our measurements are flawed, we run the risk that decisions based on these measurements be biased or distorted (Stiglitz, Sen, & Fitoussi, 2009a, 2009b).

As noted by Thorbecke (2007, p. 4), before poverty can be measured, it has at least to be understood conceptually. In this regard, our conceptual understanding of poverty has improved and deepened notably in the last four decades or so, due in large part to the seminal work of Amartya Sen and his theoretical framework of "capabilities and functionings", also called the "capability approach" (Sen, 1985, 1992, 1993, 2000a).[1] This framework represents "the most

[1] According to Sen (2000a, p. 75), "the concept of 'functionings', which has distinctly Aristotelian roots, reflects the various things a person may value doing or being. The valued functionings may vary from elementary ones, such as being adequately nourished and being free from avoidable disease, to very complex activities or personal states, such as being able to take part in the life of the community and having self-respect. A person's 'capability' refers to the alternative combinations of functionings that are feasible for her to achieve. Capability is thus a kind of freedom: the substantive freedom to achieve alternative functioning combinations (or, less formally put, the freedom to achieve various lifestyles). For example, an affluent person who fasts may have the same functioning achievement in terms of eating or nourishment as a destitute person who is forced to starve, but the first person does have a different 'capability set' than the second (the first can choose to eat well and be well nourished in a way the second cannot)".

comprehensive and logical starting point when attempting to capture the concept of poverty" Thorbecke (2007, p. 4). Under the capability approach, poverty is defined as capability deprivation, which implies, as remarked by Sen (2000a, p. 87), concentrating on deprivations that are *intrinsically* significant, unlike low income that is only *instrumentally* important; thus, poverty is seen as a multidimensional phenomenon: Human lives, as stressed by Sen (2000b, p. 18), "are battered and diminished in all kinds of different ways".

The literature that tries to cope with the fact that there are several dimensions of well-being originally focused on the measurement of multidimensional welfare and inequality. The pioneers here were Kolm (1977) and Atkinson and Bourguignon (1982).[2] These authors borrowed ideas from portfolio theory (e.g., Kolm, 1966), in particular from the literature on multivariate stochastic dominance (e.g., Levy et al., 1975).

Studies on multidimensional poverty appeared later. First, an attempt was made to apply what Pattanaik et al. (2012, p. 43) called the "column-first two-stage procedure", in which, in a first stage, the overall deprivation of society in a given domain is derived by aggregating the deprivations of the different individuals in this domain. Then, in a second stage, the overall deprivation of society is obtained by aggregating the deprivation levels of society in the different domains. This is the approach of the Human Poverty Index (UNDP, 1997). But Pattanaik et al. (2012) highlighted several shortcomings of this approach. Deriving first an aggregate distribution of achievement in various domains and, on the basis of that distribution, defining a poverty threshold and computing poverty indices is also the approach adopted by several authors using various latent variable models (see, for example, Silber, 2007, for a review of that approach, and different chapters in Kakwani and Silber, 2008, for a presentation of the application of such multivariate techniques to the analysis of multidimensional poverty).

Chakravarty et al. (1998), Tsui (2002), and Bourguignon and Chakravarty (2003) took a different route to grasp the multidimensionality of poverty. Their idea was to define a poverty line for each dimension and then to combine these different poverty thresholds and the domain-specific poverty gaps into a multidimensional poverty measure. Thus, one can assume that an individual will be poor only if he or she is poor in all attributes, or consider that, as soon as an individual is poor in one domain, he or she will be considered poor. In the first case, it is obvious that we will end up with relatively few poor people, while the second way of looking at poverty could lead to too many poor people.

[2]See also Maasoumi (1986), Tsui (1995), Maasoumi (1999) and Bourguignon (1999).

Atkinson (2003) also made an important contribution, firstly because his paper focused on the contrast between a social welfare approach and a counting approach to multidimensional poverty measurement, secondly because this paper provided a very thorough discussion on how to take into account the interaction between the various dimensions of poverty. Alkire and Foster (2011a; 2011b) extended this discussion and proposed a kind of intermediate approach between the two extreme cases known as "union" and "intersection". Their idea is to proceed in two stages. First, poverty thresholds must be defined for each dimension. Next, it is necessary to determine in how many dimensions an individual must be poor in order to be considered "multidimensionally poor". For this reason, the authors call this approach the "dual cutoff" method. Alkire and Foster (2011b) derived several multidimensional poverty measures: the traditional headcount (H), the average deprivation share across the poor (A), and the adjusted headcount ratio: $M_0 = HA$. Alkire and Foster's dual cutoff approach has, however, some shortcomings that have been criticized by diverse authors mentioned in Aaberge and Brandolini (2015) and discussed in depth in Pattanaik and Xu (2018).[3] In a recent paper, Alkire and Foster (2016) addressed some of these criticisms and introduced what they called the $M-$*gamma class* of multidimensional poverty measures that generalizes the approach of Alkire and Foster (2011b) and allows them to define a measure that takes into account inequality among the poor; however, this new class does not satisfy the dimensional breakdown property, which has proven to be important for poverty analysis. Datt (2018) also addressed some of the issues raised by those who criticized the approach of Alkire and Foster (2011b) and introduced distribution-sensitive multidimensional poverty measures that guaranteed, first, that regressive transfers in any single dimension would reduce social welfare; second, that multiple deprivations would have compound negative effects on individual and social welfare. However, Datt's (2018) approach is limited to continuous achievement variables.

A different view of multidimensional deprivation measurement was adopted by Chakravarty and D'Ambrosio (2006), who took a counting approach and proposed a measure of social exclusion, and Bossert et al. (2013), who characterized a multidimensional deprivation index in the case of discrete data. Silber and Yalonetzky (2013) proposed a general formulation[4] that includes as special cases the approaches of Alkire and Foster (2011b), Chakravarty and D'Ambrosio (2006), Rippin (2010) and Bossert et al. (2013).

[3]Pattanaik and Xu actually do not discuss the articles of Alkire and Foster (2011b) but rather review the book of Alkire et al. (2015).

[4]See also Yalonetzky (2014).

While in the papers that were just mentioned the social poverty indices are "means" or "generalized means" of the individual poverty functions, Aaberge and Peluso (2012), borrowing ideas introduced by Yaari (1987; 1988), assumed that the social poverty function was directly a function of the proportions of individuals with $1, 2, ...D$ deprivations, D being the maximal number of deprivations. Silber and Yalonetzky (2013) extended the approach of Aaberge and Peluso (2012), while Aaberge et al. (2019) extended the study of Aaberge and Peluso (2012) by providing also an empirical illustration.[5]

In the present paper, we also focus on discrete variables, in fact on dichotomous (binary) variables. The key contribution of this paper is that it introduces a rank-dependent multidimensional poverty index for multiple binary indicators using a counting approach that allows us to compute the contribution of different population subgroups to the overall level of multidimensional poverty, as well as that of the different deprivation domains. This index is, in fact, an application of Shorrocks' (1995) extension of Sen's (1976) famous uni-dimensional poverty index to the analysis of multidimensional deprivation. We call this extension *MDI*, that is, the "Multidimensional Deprivation Index". The Sen-Shorrocks index has many useful properties that turn out to have important policy implications when applied to the multidimensional case. Moreover, since the Sen-Shorrocks index can be interpreted graphically, we can compare the deprivation profiles of various countries. Thus, we also extend the *TIP curve* introduced by Jenkins and Lambert (1997; 1998a; 1998b) to the multidimensional case and call these deprivation profiles the *PUB ("Prevalence", "Unevenness" and "Breadth" of deprivation) curve.* We also highlight that the *MDI* turns out to be identical to a specific case of the Aaberge et al. (2019) deprivation measure. Finally, an empirical illustration focused on Central American countries (Guatemala, El Salvador, Honduras and Nicaragua) shows the usefulness of the *MDI* and *PUB curves.*

The paper is organized as follows. Section 2 summarizes Shorrocks' (1995) extension of the Sen (1976) index. Section 3 indicates how it is possible to define a multidimensional deprivation index (*MDI*) that is an extension of Shorrocks' (1995) approach to the case of multidimensional deprivation with dichotomous variables. Section 4 presents the properties of the *MDI.* Section 5 shows how this extension allows us to compare deprivation profiles. Section 6 provides an empirical illustration based on data from Central American countries, while Section 7 offers concluding remarks. An Appendix provides simple illustrations of the various properties of the *MDI* and of the similarity between the *MDI* and a specific case of the Aaberge et al. (2019)

[5]For more details, see also, Aaberge and Brandolini (2015).

measure.

2 On the extension of Sen's poverty index and poverty gap profiles

2.1 On Shorrocks' (1995) extension of the Sen (1976) index

Let n denote the population size, x_i the income of individual i, z the poverty line, and q the number of people with income $x_i \leq z$. Sen (1976) derived axiomatically a poverty index that is expressed as

$$P_{Sen} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2q - 2i + 1) \left(\frac{z - x_i}{z}\right) = HI + \left(\frac{q}{q+1}\right)(1 - I)\, G_P \qquad (1)$$

where H refers to the incidence of income poverty (headcount ratio), I to the intensity of poverty (the income gap ratio) and G_P to the inequality of poverty (here the Gini index of the incomes of the poor).

As stressed by Sen (1976, p. 223), the asymptotic value of P_{Sen} is $\tilde{P}_{Sen}$ where

$$\tilde{P}_{Sen} = HI + H\,(1 - I)\, G_P = H \left[1 - \frac{x_{EQ}^{poor}}{z}\right] \qquad (2)$$

where x_{EQ}^{poor} is the "equally distributed equivalent level of income" of the incomes of the poor.[6]

Takayama (1977) defined a poverty index $P_{Takayama}$ using the censored income distribution $\{x_i^*\}$, where $x_i^* = Min\{x_i, z\}$. This index is then expressed as

$$P_{Takayama} = 1 - \left(\frac{1}{n^2}\right) \frac{\sum_{i=1}^{n} \left[2\,(n-i) + 1\right] x_i^*}{\bar{x^*}} = 1 - \frac{x_{EQ,Gini}^*}{\bar{x^*}} \qquad (3)$$

where $x_{EQ,Gini}^* = \sum_{i=1}^{n} \left[\frac{2(n-i)+1}{n^2}\right] x_i^*$ is the "equally distributed equivalent level of income" with a Gini related social welfare function, while $\bar{x^*}$ is the mean of the censored income distribution.

Thon (1979) finally defined his poverty index as

[6] See, Atkinson (1970), for a definition of the notion of "equally distributed equivalent level of income", and Blackorby and Donaldson (1980) for its application to the measurement of poverty.

$$P_{Thon} = \left[\frac{2}{(n+1)nz}\right] \sum_{i=1}^{n} (z - x_i^*)(n+1-i) \tag{4}$$

Combining the concept of "equally distributed equivalent level of income among the poor" (Blackorby & Donaldson, 1980) and (3), Chakravarty (1983) defined his poverty index as

$$P_{Chakravarty} = 1 - \frac{x_{EQ}^*}{z} \tag{5}$$

x_{EQ}^* corresponding to any social welfare function, and not only to that of the Gini index, where

$$X_{EQ,Gini}^* = \sum_{i=1}^{n} \left(\frac{2n-2i+1}{n^2}\right) x_i^* \tag{6}$$

Combining (5) and (6), Chakravarty (1997) derived Shorrocks' (1995) extension of Sen's index:

$$P_{Sen-Shorrocks} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n-2i+1)\left(\frac{z-x_i^*}{z}\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n-2i+1)\left(\frac{z-x_i}{z}\right) \tag{7}$$

Shorrocks (1995) stressed that P_{Sen} in (1) is not replication invariant, is not a continuous function of individual income and does not satisfy the transfer axiom, while the $P_{Sen-Shorrocks}$ index (like $\tilde{P}_{Sen}$) is symmetric, replication invariant, monotonic, homogeneous of degree zero in z (poverty line) and x (income), normalized, continuous and consistent with the transfer axiom.

2.2 On poverty gap profiles or the so-called *TIP curve*

There has also been a graphical representation of unidimensional poverty: plot on the horizontal axis the cumulative relative frequencies of the population and on the vertical axis the cumulative values of the expression $\left(\frac{1}{n}\right) Max\{(\frac{z-x_i}{z}), 0\}$, ranking the individual by increasing income. A "poverty gap profile" (Shorrocks, 1995), also called *TIP curve* (Jenkins & Lambert, 1997; 1998a; 1998b), is then obtained. Figure 1 depicts such a curve in which

- OH refers to the proportion (q/n) of individuals who are poor.

- The slope BOD is equal to BD/OD with:

$$(BD/OD) = (AH/OD) = \left[\frac{(1/n)\sum_{i=1}^{q}\left(\frac{z-x_i}{z}\right)}{1}\right] = \frac{(q/n)\left[\sum_{i=1}^{q}\left(\frac{z-x_i}{z}\right)\left(\frac{1}{q}\right)\right]}{1} = \frac{H\sum_{i=1}^{q}\left(\frac{z-x_i}{z}\right)}{q} = H\bar{g}$$

where H is the headcount ratio, while $\bar{g}$ represents the average poverty gap among the poor.

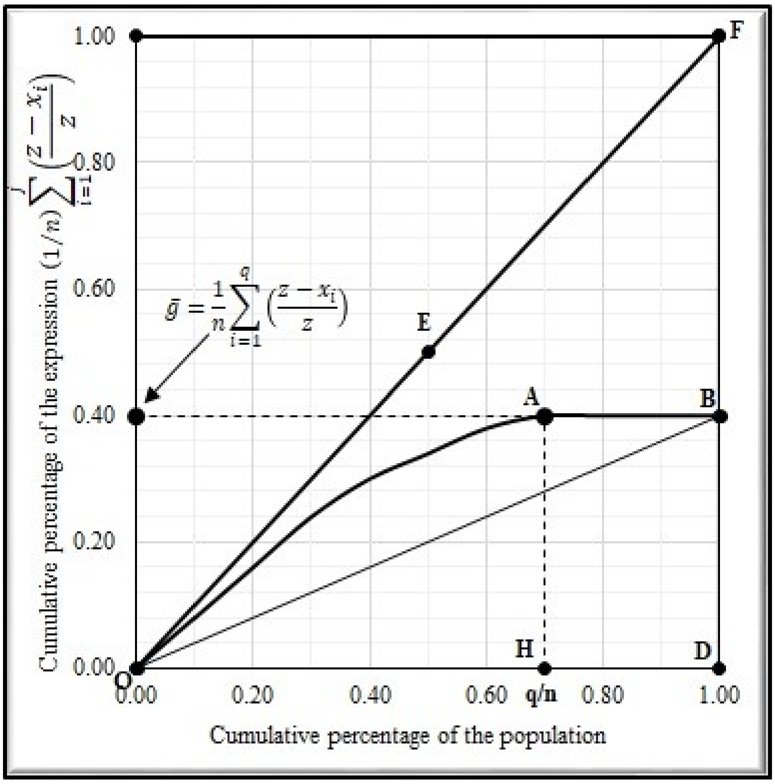

Figure 1: *TIP curve*

Shorrocks (1995) proved that the Sen-Shorrocks index is equal to twice the area below the poverty gap profile, that is, to twice the area $OABDHO$.

Clark et al. (1981, p. 519) defined the concept of "equally distributed equivalent income gap", the income gap that, if shared by every poor, would lead to the same level of welfare as the actual unequal distribution of income gaps. Defining g_i as $g_i = (z - x_i)$ and selecting a deprivation function $d(g_i) = (\frac{1}{\alpha})g_i^{\alpha}$, where α is an inequality aversion parameter ($\alpha \geq 1$), Clark et al. (1981) derived then the "equally distributed equivalent income gap" g_E where

$$g_E = \left[\left(\frac{1}{q}\right)\sum_{i=1}^{q} g_i^{\alpha}\right]^{(1/\alpha)} \tag{8}$$

In turn, Chakravarty (1983) generalized[7] the approach of Clark et al. (1983).[8]

3 Measuring multidimensional deprivation in the case of dichotomous variables

While in Section 2 we mentioned some poverty indices that were introduced in the literature on the measurement of unidimensional poverty, let us now extend the analysis to the multidimensional case.

3.1 A short review of measures of multidimensional deprivation in the case of dichotomous variables

As stressed by Dhongde et al. (2016), in the literature on multidimensional poverty there are quite a few studies using discrete data (e.g., Alkire & Foster, 2011b; Bossert et al., 2013; Lasso de la Vega, 2010), but relatively few that use binary data. Fusco and Dickes (2006) used binary data but did not propose or derive an index, but used a Rasch model. Chakravarty and D'Ambrosio (2006) axiomatically derived a social exclusion index (SE_{CD}) defined as

$$SE_{CD} = \left(\frac{1}{n}\right)\sum_{i=1}^{n} h(c_i) \tag{9}$$

where n is the number of individuals, c_i the number of goods or services that individual i does not have, while h is a function where $h(0) = 0, h^{'} > 0$ and $h^{''} \geq 0$.

Assuming a parameter $\gamma > 0$, Rippin (2010) derived the following multidimensional poverty index:

$$MP_{Rippin} = \left(\frac{1}{n}\right)\sum_{i=1}^{n} (c_i)^{\gamma+1} \tag{10}$$

Lasso de la Vega (2010) suggested a simple graphical device that allows to check the robustness of poverty rankings to changes in the identification cut-off defined in Alkire and Foster (2011b).

[7] Calling F society's deprivation function with $F = F(g_1, ..., g_i, ..., g_q)$, we get $F(g_E \cdot \mathbf{1}) = F(g_1, ..., g_i, ..., g_q)$, where $\mathbf{1}$ is a vector of ones, so that $g_E = E(g_1, ..., g_i, ..., g_q)$, where E is a particular numerical representation of F. As a general poverty index, Chakravarty (1983, p. 71) introduced the measure $P(g_1, ..., g_i, ..., g_q)$ with $P(g_1, ..., g_i, ..., g_q) = \left(\frac{q}{n}\right)\left(\frac{g_E}{z}\right)$

[8] Note that in (8) $g_E \geq \bar{g}$, where $\bar{g} = \left(\frac{1}{q}\right)\sum_{i=1}^{q} g_i$.

This partial poverty ordering is examined with respect to the multidimensional headcount ratio H and the adjusted headcount ratio M introduced by Alkire and Foster (2011b).

Assuming a parameter $r \geq 1$, Bossert et al. (2013) derived axiomatically an index of multidimensional deprivation MD_{BCD} defined as (see, Silber & Yalonetzky, 2013)[9]

$$MD_{BCD} = \left[\left(\frac{1}{n} \right) \sum_{i=1}^{n} (c_i)^r \right]^{(1/r)} \tag{11}$$

While Bossert et al. (2013) assumed variable population sizes, Dhongde et al. (2016) considered a fixed population, introduced a different concept of additive separability and made a distinction between basic attributes and non-basic attributes, where each basic attribute has priority over the class of non-basic attributes (see, Dhongde et al., 2016, for more details).

Finally, Aaberge et al. (2019) took a dual approach to multidimensional deprivation and poverty measurement and defined deprivation in society via an indicator D, where

$$D = r - \sum_{k=0}^{r-1} \Gamma(F_k) \tag{12}$$

while r is the number of possible deprivations suffered by individuals and $F_k = \sum_{h=0}^{k} f_h$, with f_h the relative frequency of those who have h deprivations. In (12) Γ is a non-negative and non-decreasing continuous function that represents the preferences of the social planner with $\Gamma(0) = 0$ and $\Gamma(1) = 1$. Since the mean number of deprivation $\bar{c}$ may be expressed as

$$\bar{c} = r - \sum_{k=0}^{r-1} F_k \tag{13}$$

We derive, combining (12) and (13), that

$$D = \bar{c} + \sum_{k=0}^{r-1} F_k - \sum_{k=0}^{r-1} \Gamma(F_k) \tag{14}$$

However, the mean difference Δ of a distribution (t) may be expressed as (see, Yitzhaki & Schechtman, 2013, p. 16)

[9]Note the similarity between MD_{BCD} and the multidimensional poverty index introduced by Bourguignon and Chakravarty (2003) for the case of continuous variables.

$$\Delta = 2\int F(t)[1 - F(t)]dt \tag{15}$$

Adapting (15) to the case of discrete data and to the distribution of deprivations, we derive that

$$\Delta_{c_i} = 2\sum_{k=0}^{r} F_k - 2\sum_{k=0}^{r}(F_k)^2 = 2\left[\sum_{k=0}^{r-1} F_k - \sum_{k=0}^{r-1}(F_k)^2\right] \tag{16}$$

where Δ_{c_i} refers to the mean difference of the deprivations, and we recall that $F_r = (F_r)^2 = 1$.

If we assume in (14) that $\Gamma(F_k) = (F_k)^2$, we conclude, using (16), that in such a case

$$D = \bar{c} + \left(\frac{1}{2}\right)\Delta_{c_i} \tag{17}$$

The case where $\Gamma(F_k) = (F_k)^2$ was indeed discussed by Aaberge et al. (2019).

3.2 Deriving a multidimensional deprivation index

Assume n individuals, J dimensions of well-being and a dichotomous variable a_{ij} equal to 1 if individual i has an achievement in domain j (e.g., if j refers to "having a good health", $a_{ij} = 1$ if individual i is in good health, to 0 otherwise). Let a_i be defined as

$$a_i = \sum_{j=1}^{J} w_j a_{ij} \tag{18}$$

where w_j is the weight of dimension j and $\sum_{j=1}^{J} w_j = 1$.

If we define d_{ij} as $d_{ij} = (1 - a_{ij})$ so that $d_{ij} = 1$ if individual i is deprived in domain j, to 0 otherwise, the weighted deprivation score (c_i) for individual i will be expressed as

$$c_i = \sum_{j=1}^{J} w_j d_{ij} \tag{19}$$

The achievement score (a_i) is a "good", so traditional tools of distributional analysis (e.g., the Lorenz or Generalized Lorenz curves) can be used. But the deprivation score (c_i) is a "bad" (see, Shorrocks, 1998), so a decrease in an individual's deprivation or inequality of deprivation scores leads to a decrease in the "aggregate deprivation".

The concept of poverty gap profile or TIP curve mentioned above can also be applied in the context of multidimensional deprivation. We define an achievement threshold t, compute the normalized achievement gaps $\left(\frac{c_i^*}{t}\right) = Max\{(\frac{t-a_i}{t}), 0\} = Max\{\frac{c_i}{t}, 0\}$ and then plot on the horizontal axis the cumulative population shares and on the vertical axis the cumulative sum of the expressions $p_i = (\frac{1}{n})(\frac{c_i^*}{t}) = (\frac{1}{n})\sum_{i=1}^{q}(\frac{c_i}{t})$, the c_i^*'s being ranked by decreasing values, an ascending curve is obtained whose slope is non-increasing and equal to 0 when we reach the $(n-q)$ individuals with no deprivation (there are q individuals with at least one deprivation). The curve is similar to the one in Figure 1, but now

- OH refers to the *prevalence* P of deprivation [proportion $P = (q/n)$ of individuals having some deprivation].

- The slope BOD equals $(BD/OD) = \left[\frac{(1/n)\sum_{i=1}^{n} c_i^*}{1}\right] = \frac{(1/n)\left[\sum_{i=1}^{q} c_i\right]}{1} = \left(\frac{q}{n}\right)\left(\frac{\sum_{i=1}^{q} c_i}{q}\right) = \left(\frac{q}{n}\right)\bar{c}_q$ where $\bar{c}_q$ represents the average percentage of deprivations among those who have at least one deprivation; $\bar{c}_q$ could be labeled the *breadth (B)* of deprivation.

In Figure 1, the curvature of the OA curve indicates the extent of inequality among those deprived in at least one dimension or the *unevenness (U)* of deprivation.

Given that the "deprivation curve" (OAB) takes into account the *prevalence (P)*, the *unevenness (U)* and the *breadth (B)* of deprivation, we suggest to call it the *PUB curve*, an adaptation of the *TIP curve* to multidimensional deprivation with dichotomous variables.[10]

Shorrocks (1995) showed that twice the $OABDHO$ area is equal to the Sen-Shorrocks index, so we will consider twice this area as a "Multidimensional Deprivation Index" (or MDI for short).

If we adapt expression (7) to the case of multidimensional deprivation, we may write that

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1)\left(\frac{c_i^*}{t}\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n - 2i + 1)\left(\frac{c_i}{t}\right) \quad (20)$$

With a union approach (an individual is deprived even if in only one domain), $t = 1$ and then

[10] Lasso de la Vega (2010) had also introduced deprivation curves derived from deprivation counts, what she called the *FD* and the *SD* curves. These curves are however different from the *PUB curve* introduced in this paper.

$$MDI_{union} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1)\, c_i^* = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n - 2i + 1)\, c_i \tag{21}$$

Using (20), the contribution ($Cont_i$) of individual i to the overall deprivation is expressed as

$$Cont_i = 2\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{c_i}{t}\right)\left[\left(\frac{2n+1}{2}\right) - i\right] \tag{22}$$

Adapting Shorrocks' (1995) equation (12) to multidimensional deprivation, we may write that

$$MDI = \bar{c}\,(1 + G_{c_i}) = \bar{c}\left[1 + \left(\frac{c_{EQ} - \bar{c}}{\bar{c}}\right)\right] = c_{EQ} = \bar{c} + (\frac{1}{2})\Delta_{c_i} \tag{23}$$

where $\bar{c}$ and G_{c_i} are respectively the average level of deprivation and the Gini index of the deprivation scores in the whole population (including those who have no deprivation) and $\Delta_{c_i} = 2\bar{c}G_{c_i}$ is the mean difference of the deprivations.

We may observe that expressions (17) and (23) are identical so that the MDI is a specific case of the deprivation measure of Aaberge et al. (2019), that where $\Gamma(F_k) = (F_k)^2$.

Calling c_{EQ} the "equally distributed equivalent deprivation score",[11] we rewrite (23) as

$$MDI = \bar{c}\,(1 + G_{c_i}) = \bar{c}\left[1 + \left(\frac{c_{EQ} - \bar{c}}{\bar{c}}\right)\right] = c_{EQ} \tag{24}$$

Instead of using the traditional Gini index G_{c_i} in (23) and (24) one can also use the generalized Gini index that was introduced by Donaldson and Weymark (1980) and apply it to the deprivation scores. The "equally distributed equivalent deprivation score" $c_{EQ,GEN}$ in such a case will use the concept of "ill-fare ranking" (Donaldson & Weymark, 1980) so that

$$c_{EQ,GEN} = \sum_{i=1}^{n} \left(\frac{i^{\beta} - (i-1)^{\beta}}{n^{\beta}}\right) c_i \tag{25}$$

with $0 \leq \beta \leq 1$ and evidently $c_1 \geq ... \geq c_q \geq ...0$.

[11]It is well known that the Gini index of incomes I_G, like several other income inequality indices that can be related to a welfare function, may be expressed as $I_G = (\frac{\bar{y} - y_E}{\bar{y}})$, where $\bar{y}$ refers to the average income and y_E to Atkinson's (1970) "equally distributed equivalent level of income". While income is a "good", deprivation is a "bad" so that the Gini index of the deprivation scores is defined as $G_{c_i} = \frac{(c_{EQ} - \bar{c})}{\bar{c}}$.

3.3 Estimating the contribution of different population subgroups to the *MDI*

Assume K population subgroups, each subgroup k with n_k individuals. Using (20), we write

$$MDI = \left(\frac{1}{n}\right)^2 2 \sum_{k=1}^{K} \sum_{i \in k} \left(\frac{c_i}{t}\right) \left[\left(\frac{2n+1}{2}\right) - i\right] \tag{26}$$

i being the ranking of the individual in the whole population and not in his/her subgroup.

The contribution C_k of population subgroup k to multidimensional deprivation is hence

$$C_k = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) 2 \sum_{i \in k} \left(\frac{c_i}{t}\right) \left[\left(\frac{2n+1}{2}\right) - i\right] \tag{27}$$

3.4 Making assumptions concerning the weight of the different deprivation domains

Let j refer to a given deprivation domain with $j = 1$ to J. Combining (19) and (20), we derive

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \sum_{j=1}^{J} \frac{w_j d_{ij}}{t} (2n - 2i + 1) \tag{28}$$

so that the contribution $CONTR_j$ of deprivation domain j to the overall deprivation becomes

$$CONTR_j = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} w_j \left(\frac{d_{ij}}{t}\right) (2n - 2i + 1) \tag{29}$$

There are quite a few possibilities regarding the choice of the weights (w_j) of the different dimensions. However, in a recent paper, Dutta et al. (2021) have shown that endogenous (data driven) weights violate the key properties of poverty indices, namely monotonicity and subgroup consisteny. Therefore, they have recommended using exogenous weights, the simplest case being the one in which all deprivation domains have the same weight. We will make this assumption so that we rewrite (29) as

$$CONTR_j = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \left(\frac{1}{J}\right) \left(\frac{d_{ij}}{t}\right) (2n - 2i + 1) \tag{30}$$

It is then clear that the MDI is factor decomposable. Using Alkire and Foster's terminology (Alkire & Foster, 2011b), we can state that the MDI fully satisfies the "dimensional breakdown" property. It is also possible to standardize the MDI as shown in Appendix A.

4 Properties of the *MDI*

As emphasized above, the MDI is simply the Sen-Shorrocks poverty index applied to the weighted deprivation scores c_i. Therefore, all the properties of the Sen-Shorrocks index stated by Shorrocks (1995) and mentioned previously also hold for the MDI.

Alkire and Foster (2016) have stated that the properties of multidimensional poverty methodologies can be classified into three categories: *invariance*, *subgroup* and *dominance properties*.

Invariance properties include those of *symmetry*, *replication invariance*, *deprivation focus* and *poverty focus*.

4.1 Invariance properties

Symmetry

The reference here is to permutations of achievement vectors across individuals. As stressed by Shorrocks (1995), the Sen-Shorrocks poverty index has this property.

Population replication

Assume a "cloning" of the whole population so that the total population and the number of deprived individuals are now respectively equal to (λn) and (λq), with λ an integer greater than 1. We assume no change in the number of dimensions. In addition, any deprived individual (i) with a deprivation score c_i will be replaced by λ individuals with this deprivation score c_i. Here again, Shorrocks (1995) stated that such a property holds for the Sen-Shorrocks poverty index.

Poverty focus

This assumption says that an increment in the achievement of a non-deprived person, that is, of an individual who is not deprived in any dimension, will not affect the value of the multidimensional deprivation index (MDI). This should be clear from equation (20), since the MDI is only a function of the deprivation of the deprived individuals.

Deprivation focus

This property assumes that the multidimensional deprivation index (MDI) will be invariant to an increment in a non-deprived achievement. It is easy to prove this property as well, since if an individual i improves his/her achievement in a dimension j in which he/she was not deprived, the value of the dichotomous variable d_{ij} will not vary and remain equal to 0.

4.2 Subgroup properties

Alkire and Foster (2016) have mentioned the properties of *subgroup consistency* and *subgroup decomposability*.

Subgroup decomposability

The expression for the contribution of subgroup k to the overall deprivation (MDI) appears in expression (27) in Section 3.2. Combining (26) and (27), we conclude that

$$MDI = \sum_{k=1}^{K} C_k \tag{31}$$

We can therefore compute the contribution of each subgroup to the overall level of deprivation. However, note that C_k in (27) is not identical to what would be the definition of an MDI limited to group k. This is so because the coefficient $\left[\left(\frac{2n+1}{2}\right) - i\right]$ associated to the deprivation component $\left(\frac{c_i}{t}\right)$ of individual i depends on the rank of individual i in the whole population, and not in subgroup k. A subgroup decomposable deprivation index would be expressed as the sum of a between and a within groups deprivations. But this is not what (27) is expressing. Therefore, we cannot conclude that the multidimensional deprivation index (MDI) is subgroup decomposable in the traditional interpretation of such a breakdown. This is also the case of the Gini index, since it is well known that, as soon as there is some overlap between the population subgroups, the decomposition of the Gini index will include three components: a between and a within groups inequality but also a residual, which has been shown to be a measure of the overlap between the different distributions (see, for example, Silber, 1989).

It is however possible to take an alternative view of the breakdown of the MDI by population subgroups. To derive such an alternative decomposition, we borrow ideas from the literature on alternative decompositions of the Gini index. Deutsch and Silber (1999) have indicated that there is no unique way of decomposing inequality by population subgroups. In particular, they have mentioned a decomposition of the Gini index, originally proposed by Lerman and Yizthaki (1991) and Sastry and Kelkar (1994), where the Gini index turns out to be the sum of a between and within groups components, but these two components are not defined in the traditional way. The idea is to keep the original ranking of the individuals, when computing these between and within group components. This idea may be also applied to the breakdown of the MDI into a between and a within groups components.

The alternative between groups MDI is then defined as

$$MDI_{BETWEEN}^{Alternative} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1)\left(\frac{\bar{c_i}}{t}\right) \tag{32}$$

where i refers to the original rank of an individual and $\bar{c_i}$ refers to the average deprivation level in the population subgroup to which individual i belongs.

The alternative within groups component is then expressed as

$$MDI_{WITHIN}^{Alternative} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1)\left[\frac{(c_i - \bar{c_i})}{t}\right] \tag{33}$$

In Appendix B, we give a simple empirical illustration of what we called the traditional and the alternative decompositions of the MDI. Figures 3 and 4 give also a graphical representation of the two decompositions.

In short, when using the alternative approach, it is possible to affirm that the MDI is decomposable by population subgroups.

Subgroup consistency

Shorrocks (1995, p. 1226) has stressed that, like the Sen poverty index P_{Sen}, the Sen-Shorrocks poverty index ($P_{Sen-Shorrocks}$) is not subgroup consistent, but "it is an ideal measure of poverty in all other respects". Since the MDI is equivalent to the $P_{Sen-Shorrocks}$ index, but applied to multidimensional deprivation, we conclude that the MDI is not subgroup consistent.

4.3 Dominance

Alkire and Foster (2016) have included two properties here. First, there is the concept of *Weak Monotonicity* according to which an increase in the achievement of an individual cannot increase deprivation. Then, there is the notion of *Weak Rearrangement*, which requires that a progressive transfer among the deprived individuals, which is the consequence of an "association-decreasing rearrangement", cannot increase deprivation.

Monotonicity

Shorrocks (1995) stated that the index $P_{Sen-Shorrocks}$ is monotonic. We can therefore conclude that the multidimensional deprivation index (MDI) has the property of monotonicity.

Transfers

Let us first state that in the context of uni-dimensional poverty measurement Shorrocks (1995) stressed that the $P_{Sen-Shorrocks}$ index is consistent with the transfer axiom. When applying this property to multidimensional deprivation analysis, we can therefore conclude that if, within a given deprivation domain j, a transfer takes place from a more to a less deprived individual, assuming no change in the ranking of the individuals, the MDI will decrease. More precisely, assume that originally individual i as a whole was more deprived than individual m and was deprived in domain j while individual m was not. After the "transfer" individual i remains more deprived than individual m, but he/she has one deprivation less, while individual m has one more deprivation than originally. In such a case the MDI will decrease.

The same kind of reasoning applies when a transfer takes place between individuals and across domains. Assume, for example, that individual h has n_h deprivations and that individual i has n_i deprivations with $n_h > n_i$, that individual h is deprived in domain j but not in domain k and individual i in domain k but not in domain j. If, for some reason, a change occurs such that individual h is not deprived any more in domain j while individual i who was deprived in domain k becomes also deprived in domain j. Suppose, however, that, after such a "transfer" of deprivations, individual h has still more deprivations than individual i. Assuming that all the domains have the same weight, it is easy to observe, using (20), that the MDI will decrease.

Given that in the formulation of the MDI in (20), which refers to the case of equal weights, only the number of deprivations of each individual is taken into account, regardless of in which domains these deprivations take place, the notion of "*Weak Dimensional Rearrangement among the deprived individuals*", which was discussed by Alkire and Foster (2016), is not relevant.

Instead of analyzing the impact of a transfer of deprivations between two individuals h and i, let us assume that these two individuals switch their deprivations. In other words, using the example given above, we would observe that in the new situation individual h is deprived in domain k but not in domain j and individual i is deprived in domain j but not in domain k.

Clearly, this switch will not affect the number of deprivations of each individual and, hence, there will be no change in the value of the MDI.

In defining the MDI in (20), which refers to the case of equal weights for the different deprivation domains, we make the assumption that the various deprivation domains are perfect substitutes. The situation is different when examining the case of unequal weights. It should be clear that, even in the case where the various dimensions have different weights, a transfer of deprivations between two individuals of the kind described above, whether it takes place within a given domain or across domains, will lead to a decrease in the MDI, as long as the ranking of the individuals is not affected by the number of deprivations they suffer from. However, when the deprivation domains have not the same weight, the switch of deprivations between two individuals and two domains with unequal weights, will lead either to an increase or a decrease in the value of the MDI, depending on the assumption made about the weights of domains j and k.

5 Comparing deprivation profiles and comparing *MDI* indices

The ordinal approach to uni-dimensional poverty analysis seems to have been originally introduced by Spencer and Fisher (1992). Jenkins and Lambert (1997, p. 317) then introduced the concept of TIP ("Three I's of Poverty") curves, these three I's referring respectively to the *incidence*, *intensity* and *inequality* of poverty. Subsequently, Jenkins and Lambert (1998b, p. 47) stated in their Theorem 3 that "given any two income distributions x and y and poverty lines z_x and z_y, TIP dominance of the normalized poverty gap distribution Γ_y over the normalized poverty gap distribution Γ_x is necessary and sufficient to ensure $Q(x \mid k, z_x) \leq Q(y \mid k, z_y)$ for all $k \in (0, 1]$ and for all povety measures $Q \in \mathbf{Q}$", the latter being replication invariant and increasing Schur-convex functions of the normalized gaps. These deprivation profiles or TIP curves may naturally be used when adopting the $P_{Sen-Shorrocks}$ rather than the P_{Sen} index, as shown in Shorrocks (1995).

The MDI introduced in the present paper is an adaptation of the $P_{Sen-Shorrocks}$ index to the case of multidimensional deprivation. Moreover, we have shown previously that our *PUB curve* is a simple adaptation of the notion of *TIP curve* to the multidimensional case, assuming that deprivation in a given domain is only measured via dichotomous variables. We can therefore apply the theorem of Jenkins and Lambert (1998b) stated previously, provided the deprivation profiles of the distributions we compare, do not intersect.[12]

6 An simple empirical illustration

In this section, we present an empirical illustration of the *PUB curve*, the MDI and its decomposition by deprivation indicator, using data from four Central American countries, namely, Guatemala, El Salvador, Honduras, and Nicaragua, and taking as a reference the work by Espinoza-Delgado and Silber (2018, 2021). To estimate multidimensional poverty in these Central American countries, we used data from the Guatemala National Survey of Living Conditions (2014) (GUA-ENCOVI2014), the El Salvador Multipurpose Household Survey (2016) (ELS-EHPM2016), the Honduras Multipurpose Household Survey (2013) (HON-EPHPM2013), and the Nicaragua National Household Survey on Living Standards Measurement (2014) (NIC-EMNV2014), which are nationally representative. In our exercise, we focus on individuals who are between 18 and 59 years old, are identified as household members and completed a full interview; in other words, we use the individual, rather than the household, as the unit of analysis and focus on the adult members of the households, approximately 50% of the population in the countries studied (from a low of 47.7% in Honduras up to a maximum of 59.3% in El Salvador).

Regarding the empirical design of the MDI, we considered five deprivation dimensions (education, employment, water and sanitation, energy and electricity, and the quality of the dwelling) with ten indicators, which are certainly among the most significant aspects of individual well-being (Stiglitz et al., 2009a, 2009b). The specific indicators chosen for each of the five dimensions and the corresponding deprivation definitions are presented in Table 1; this table also shows the weighting structure that we used: equal-nested weights.

The PUB curve: prevalence (P), unevenness (U) and deprivation breadth (B) curve

We assumed that the threshold t was equal to 1. Figure 2 displays the *PUB curve* for Guatemala, El Salvador, Honduras, Nicaragua, and Central America as a whole; in this figure, the cumulative population frequencies are plotted on the X-axis, while the cumulative values of $\left(\frac{1}{n}\right)\sum_{i=1}^{n}\left(\frac{c_i}{1}\right)$ are plotted on the Y-axis.

[12] In a recent paper, Azpitarte et al. (2020) have introduced fundamental conditions whose fulfilment is both necessary and sufficient to ensure that poverty comparisons are robust to changes in individual poverty functions, dimensional weights and poverty cut-off. As stated by the authors, these conditions may be cumbersome when the number of variables is large. This is why they have also derived conditions whose fulfilment is necessary, but insufficient for robust first- and second-order poverty comparisons. The extension of the Sen-Shorrocks index to multidimensional poverty proposed in the present paper might be a simpler way of analyzing dominance.

Table 1: Dimensions [in parenthesis the related Sustainable Development Goal (SDG)], indicators, weights, and deprivation cut-offs

Dimensions	Indicators	Weights (%)	Deprivation indicators: He / She is deprived if He / She...
1. Education (Goal 4 of the SDGs)	1.1. Schooling achievement	20	has not completed lower secondary school (nine years of schooling approximately).
2. Employment (Goal 8 of the SDGs)	2.1. Employment status	20	is unemployed, employed without pay, or a discouraged worker or a domestic worker or an unpaid care worker who reported that he/she ”did not have a job” but was available to work.
3. Water and sanitation (Goal 6 of the SDGs)	3.1. Improved water source	10	does not have access to an improved water source or has access to it, but out of the house and yard/plot.
	3.2. Improved sanitation	10	only has access to an unimproved sanitation facility (a toilet or latrine without treatment or a toilet flushed without treatment to a river or a ravine) or to a shared toilet facility.
4. Energy and electricity (Goal 7 of the SDGs)	4.1. Type of cooking fuel	10	is living in a household which uses wood and/or coal and/or dung as main cooking fuel.
	4.2. Access to electricity	10	does not have access to electricity.
5. Quality of dwelling (Goal 11 of the SDGs)	5.1. Housing materials	5	is living in a house with dirt floor and/or precarious roof (waste, straw, palm and similar, other precarious material) and/or precarious wall materials (waste, cardboard, tin, cane, palm, straw, other precarious material).
	5.2. People-per-bedroom	5	has to share a bedroom with two or more people.
	5.3. Housing tenure	5	is living in an illegally occupied house or in a borrowed house.
	5.4. Assets	5	does not have access to more than one durable good of a list that includes: Radio, TV, Refrigerator, Motorbike, Car.

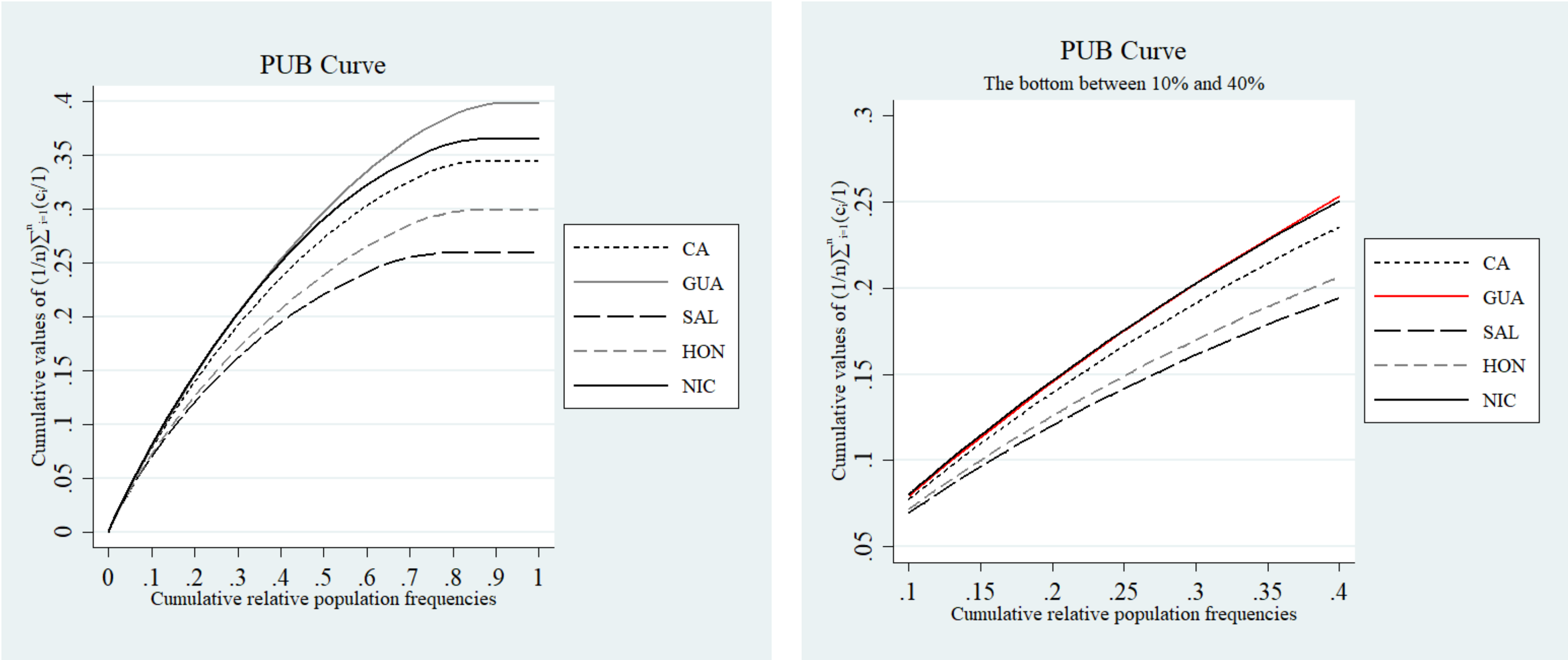

Figure 2: Resulting "*PUB curve*" for Central American as a whole (CA), Guatemala (GUA), El Salvador (SAL), Honduras (HON), and Nicaragua (NIC). *Source*: Authors' estimates based on GUA-ENCOVI2014, ELS-EHPM2016, HON-EPHPM2013, and NICEMNV2014.

Table 2: Absolute and relative contributions of each indicator to the overall MDI. *Sources*: Authors' estimates based on GUA-ENCOVI2014, ELS-EHPM2016, HON-EPHPM2013, and NIC-EMNV2014.

Guatemala

Contrib.	Education	Employment	Water	Sanitation	Energy	Electricity	Housing	Overcrowding	Housing tenure	Assets	MDI
Absolute	0.2645	0.0664	0.0471	0.1107	0.1444	0.0360	0.0309	0.0495	0.0103	0.0357	0.7956
Relative	33.2%	8.3%	5.9%	13.9%	18.2%	4.5%	3.9%	6.2%	1.3%	4.5%	100.0%

El Salvador

Contrib.	Education	Employment	Water	Sanitation	Energy	Electricity	Housing	Overcrowding	Housing tenure	Assets	MDI
Absolute	0.1736	0.0726	0.0417	0.0849	0.0204	0.0256	0.0184	0.0450	0.0180	0.0167	0.5168
Relative	33.6%	14.0%	8.1%	16.4%	3.9%	4.9%	3.6%	8.7%	3.5%	3.2%	100.0%

Honduras

Contrib.	Education	Employment	Water	Sanitation	Energy	Electricity	Housing	Overcrowding	Housing tenure	Assets	MDI
Absolute	0.2365	0.0644	0.0247	0.0466	0.1100	0.0242	0.0177	0.0439	0.0056	0.0233	0.5969
Relative	39.6%	10.8%	4.1%	7.8%	18.4%	4.1%	3.0%	7.3%	0.9%	3.9%	100.0%

Nicaragua

Contrib.	Education	Employment	Water	Sanitation	Energy	Electricity	Housing	Overcrowding	Housing tenure	Assets	MDI
Absolute	0.2247	0.0786	0.0676	0.0860	0.1051	0.0265	0.0394	0.0535	0.0157	0.0332	0.7303
Relative	30.8%	10.8%	9.3%	11.8%	14.4%	3.6%	5.4%	7.3%	2.2%	4.5%	100.0%

Central America as a whole

Contrib.	Education	Employment	Water	Sanitation	Energy	Electricity	Housing	Overcrowding	Housing tenure	Assets	MDI
Absolute	0.2342	0.0693	0.0448	0.0876	0.1066	0.0298	0.0272	0.0481	0.0117	0.0290	0.6883
Relative	34.0%	10.1%	6.5%	12.7%	15.5%	4.3%	3.9%	7.0%	1.7%	4.2%	100.0%

Note: surveys weights used.

Overall, the left side of Figure 2 suggests that in the Central American region, the highest and lowest levels of multidimensional poverty are found in Guatemala and El Salvador, respectively. The *PUB curve* of Honduras dominates that of El Salvador, so that multidimensional poverty in the former country is always higher than in the latter, regardless of the population decile we choose. The cases of the Guatemalan and Nicaraguan curves are interesting. Figure 2 shows that the Nicaraguan curve crosses the Guatemalan curve once from above around the 25% point on the horizontal axis (see the right side of the figure), suggesting that overall multidimensional poverty is higher in Guatemala than in Nicaragua only from this point on, i.e., the poorest of the poor are in Nicaragua.

As discussed when presenting the MDI, one of the key properties for policy design that our index satisfies is the fully factorial decomposability property. Table 2 illustrates this decomposition for the case of Guatemala, El Salvador, Honduras, and Nicaragua, as well as for Central American as a whole. This table presents the absolute and relative contributions to the overall estimate of multidimensional poverty of each of the ten indicators used to measure multidimensional poverty in Central America; the overall estimates are shown in the last column of the table. The table indicates that in Central America, education is the largest contributor to multidimensional poverty; deprivations in this dimension account for one-third of the estimated MDI in each of the countries.

7 Concluding comments

This paper has introduced a multidimensional deprivation index (MDI) that is an adaptation to the multidimensional case of the Sen-Shorrocks index of unidimensional poverty. It turns out that this index is a particular case of a measure of multidimensional deprivation recently introduced by Aaberge et al. (2019). In addition, by linking the MDI to the Sen-Shorrocks index, we were able to derive a simple graphical representation that we called *PUB curve* (prevalence, unevenness and breadth of deprivation curve), which is an adaptation to the multidimensional case of the TIP curve of Jenkins and Lambert (1997). As a consequence, it is possible to compare the deprivation profiles of two or more countries or of a country during various periods and to derive dominance relationships.

The main advantage of the MDI is that it can be simply broken down by deprivation domain as well as by population subgroup, although it is not a subgroup consistent index, but "it is an ideal measure of poverty in all other respects" (Shorrocks, 1995, p. 1226). These two decomposition properties have important implications because they allow policy makers to detect the deprivation domains that require special attention and to focus their attention on the population subgroups which should be helped in priority. The empirical illustration of the paper, which looked at four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua), allowed us to conclude that education is the largest contributor to multidimensional deprivation since it accounts for one-third of the MDI in each of the countries.

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Appendix A: Standardizing the *MDI*

Computing the minimal and maximal values of the multidimensional deprivation index (MDI)

Minimal value of the MDI

The determination of the minimal value of the MDI depends on the conditions under which we want to define this minimum. If we do not impose any restrictions, then evidently the minimum will be equal to zero and will be reached when no one is deprived.

We can also assume that only the number (q) of deprived people is fixed, in which case the minimum will be reached when each deprived person has only one deprivation.

It is, however, more interesting to ask what this minimal value will be, assuming a given average $\bar{c} = \frac{\sum_{i=1}^{n} c_i}{n}$ of the individual deprivation scores in the whole population and a given percentage (q/n) of deprived individuals. In other words, in Figure 1, we assume a given value of the angle BOD since the tangent of this angle is equal to the average deprivation score among the poor times the headcount ratio. We recall that A is the point at which the PUB curve becomes horizontal while on the horizontal axis we note that $OH = (q/n)$, which is the percentage of deprived individuals. It should then be clear that, given the assumptions made, the MDI, which is equal to twice the area under the deprivation curve, will be minimal when the OAB curve is a straight line, that is, when all the individuals have the same deprivation score $\bar{c} = \frac{\sum_{i=1}^{n} c_i}{n}$.

In such a case (see Figure 1), $OD = 1, BD = \frac{\sum_{i=1}^{n} c_i}{n} = \bar{c}$ so that $BD/OD = \bar{c}$ and the area OBD is equal to $\left(\frac{1}{2}\right)\bar{c}$.

In the case where the number of deprivations of each individual is an integer number and all the dimensions have an equal weight, we may not have a straight line (OB) to characterize minimum deprivation. Take, for example, the case where we have 5 dimensions, three individuals, and the total number $\sum_{i=1}^{q} c_i$ is equal to 13. Then, clearly, the minimum deprivation will take place when the first individual has five deprivations, the second four and the third four deprivations. More details and other illustrations are given in Appendix A.3.

Maximal value of the MDI_G

The determination of the maximal value of the MDI_G also depends here on the conditions under which we want to define this maximum. If we do not impose any restrictions, then evidently the maximum will be reached when each individual is deprived and the number of his/her deprivations is equal to the number of dimensions.

But here again it will be more interesting to ask what this maximal value will be, assuming a given average $\bar{c}$ (or a given sum $\sum_{i=1}^{n} c_i$) of the individual deprivation scores.

Let us, as previously, limit ourselves to the case when the number of deprivations of an individual is an integer number. Take again the case where we have 5 dimensions, three individuals and the total number $\sum_{i=1}^{q} c_i$ is equal to 13. Then, clearly, maximum deprivation will take place when the first individual has five deprivations (the maximum number he/she can have), the second five and the third three deprivations. More details and other illustrations are here also given in Appendix A.3.

Standardizing the multidimensional deprivation index

Let $MaxMDI$ and $MinMDI$ refer respectively to the maximal and minimal values of the multidimensional deprivation index (MDI), for a given number J of deprivation dimensions and a given sum $\sum_{i=1}^{q} c_i$ of the total number of deprivations in the population of deprived people.

The standardized multidimensional deprivation index ($MDI_{standard}$) may then be expressed as

$$MDI_{standard} = \left(\frac{MDI - Min\{MDI\}}{Max\{MDI\} - Min\{MDI\}} \right) \tag{34}$$

We then observe that $0 \leq MDI_{standard} \leq 1$.

Appendix B: The decomposition of the *MDI* by population subgroups

Assume a population of four individuals. Three of them have a certain number of deprivations and one is without any deprivation so that $n = 4$ and $q = 3$. Suppose that there are 5 domains of deprivation ($j = 1$ to 5). Individual 1 is deprived in domains 1, 2, 4, 5 so that ($c_1 = (4/5)$), individual 2 in domains 3 and 4 ($c_2 = (2/5)$) and individual 3 in domain 5 ($c_3 = (1/5)$). Individual 4 has no deprivation. Suppose that individuals 1 and 3 belong to group A and individuals 2 and 4 to group B. Let us also assume that the threshold t is equal to 1. Finally define p_i as that $p_i = (1/n)c_i = (1/4)c_i$. Figure 3 illustrates this case.

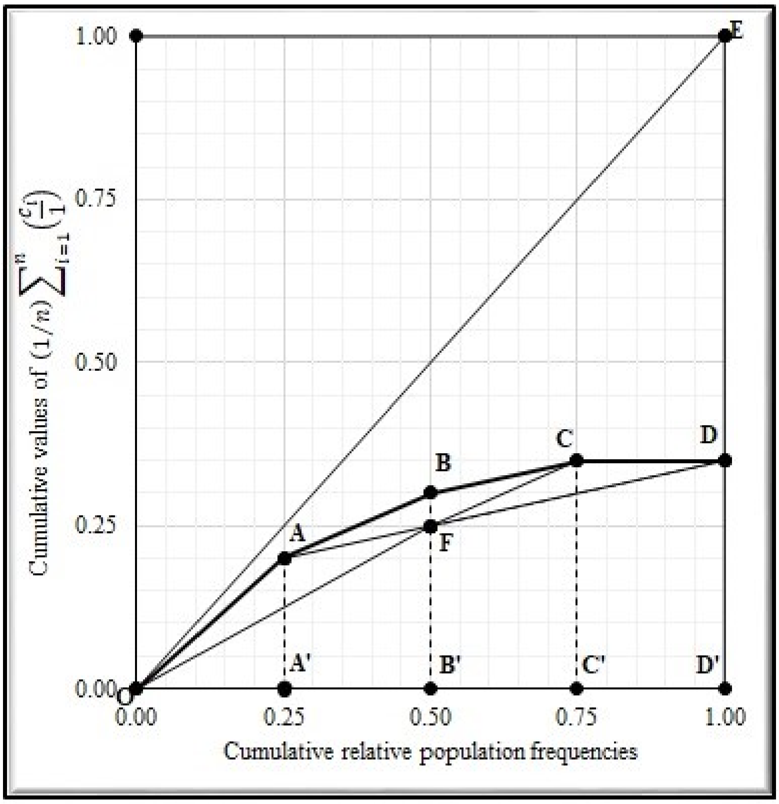

Figure 3: Illustration of the decompostion of the MDI by population subgroups

Using (20) the MDI is expressed as

$$MDI = \left(\frac{1}{16}\right)\{[(7)(0.8)] + [(5)(0.4)] + [(3)(0.2)]\} = \frac{(5.6+2+0.6)}{16} = \frac{8.2}{16}$$

Using (27) we then derive that the contributions C_A and C_B of groups A and B are expressed as

$$C_A = (1/16)\{[(7)(0.8)] + [(3)(0.2)]\} = \frac{6.2}{16}$$

$$C_B = (1/16)\{[(5)(0.4)]\} = \frac{2}{16}$$

It is easy to observe that, as expected, the sum of these two contributions is equal to $\frac{6.2+2}{16} = \frac{8.2}{16}$, which is the value of the MDI for the whole population.

The graphical representation of a traditional decomposition

In Figure 3 the curve $OABCD$ represents what we previously called the PUB curve. The line OE is the deprivation curve that would be obtained if everyone had the same and maximal level of deprivation, namely (5/5) so that the height ED' is, as expected, equal to $4(1/4)(5/5) = 1$. It is easy to check that the heights AA', BB', CC' and DD are respectively equal to 0.2, 0.3, 0.35 and 0.35 and that the areas OAA', $AA'B'B$, $BB'C'C$ and $CC'D'D$ are respectively equal to 0.025, 0.0625, 0.08125 and 0.0875. The sum of these 4 areas which corresponds to the area $OABCDD'C'B'A'O$ is then equal to 0.25625. Twice this sum gives us $0.5125 = (8.2/16)$, which is, as expected and shown previously, the value of the MDI when all the domains have the same weight.

Given that individuals 1 and 3 belong to group A and individual 2 and 4 to group B, it is easy to check that the average number of deprivations in group A is $(4 + 2)/2 = 3$ and in group B it is $((2 + 0)/2) = 1$. We can therefore draw in Figure 3 a broken curve OFD. On the section OF, the height of point F corresponds to the total deprivation in group A, which includes individuals 1 and 3 and hence it is equal to $[(1/4)(4/5)] + [(1/4)(1/5)] = (5/20) = 0.25$. Similarly, the difference between the height of point D and that of point F corresponds to the deprivation in group B and is hence expressed as $[(1/4)(2/5)] + [(1/4)(0/5)] = (2/20) = 0.1$. The height of point D is therefore $0.25 + 0.1 = 0.35$. The area below the curve $OFDD'O$ is therefore, computed as $[(1/2)(0.5)(0.25)] + \{(1/2)(0.5)[0.25 + 0.35]\} = 0.0625 + 0.150 = 0.2125$. Twice this area, that is, 0.425, is hence the between groups A and B components of multidimensional deprivation.

We can also compute the within groups A and B components of multidimensional deprivation. The within group A deprivation is evidently the area OAF while that within group B is the area FCD. Now $OAF = [(OAA') + (AA'B'F)] - (OFB')$ with $OAA' = [(1/2)(0.25)(0.2)] = 0.025$; $AA'B'F = [(1/2)(0.25)(0.2{+}0.25)] = 0.05625$; $OFB' = [(0.5)(0.5)(0.25)] = 0.0625$. We therefore derive that the area OAF is equal to $(0.025 + 0.05625) - 0.0625 = 0.01875$. Twice this number gives us the within group A multidimensional deprivation and it is equal to 0.0375.

The within group B deprivation is given by the triangle FCD whose area is equal to $[(FB'C'C + CC'D'D) - FB'D'D]$. But $FB'C'C = (1/2)(0.25)(0.25 + 035) = 0.075$; $CC'D'D = (0.250.35) = 0.0875$; and $FB'D'D = (1/2)(0.5)(0.25 + 0.35) = 0.15$. The area FCD is hence equal to $(0.075 + 0.0875) - 0.15 = 0.0125$. Twice this area is therefore equal to the within group B multidimensional deprivation, that is, to 0.025.

Let us now compute the area $ABCF$ that corresponds to the overlap between group A and group B. We may write that $ABCF = (AA'B'B + BB'C'C)$ - $(AA'B'F + FB'C'C)$. $AA'B'B = (0.5)(0.25)(0.2{+}0.3) = 0.0625$; $BB'C'C = (0.5)(0.25)(0.3 + 0.35) = 0.08125$; $AA'B'F = (0.5)(0.25)(0.2 + 0.25) = 0.05625$; $FB'C'C = (0.5)(0.25)(0.25 + 0.35) = 0.075$. Therefore, $ABCF = (0.0625 + 0.08125)$ - $(0.05625 + 0.075) = 0.0125$. Twice this area will be the overlap component of the MDI, and it is equal to 0.025.

The sum of the three components (between groups, within groups and overlap deprivation) is then equal to $(0.425 + 0.0375 + 0.025 + 0.025) = 0.5125 = (8.2/16) = MDI$.

The graphical representation of an alternative decomposition of the MDI

Figure 4 gives a graphical representation of this alternative decomposition.

As in Figure 3, the curve $ABCD$ represents the actual PUB curve, and it is drawn by ranking the individuals by decreasing level of deprivation This ranking will be kept when drawing the deprivation curve that would be observed if each individual's deprivation was the average deprivation of the group to which he/she belongs. We saw previously that the average deprivation in group A, which includes individuals 1 and 3, is $(4+1)/2 = 2.5$ while the average deprivation in group B is $(2+0)/2 = 1$. Keeping the original ranking of the individual we conclude that the height of point A'' which corresponds to this deprivation of individual 1 will be $(1/4)(2.5/5) = (2.5/20) = 0.125$. To reach the second point (B'') on this "alternative average deprivation curve", we add to the height of point A' the average deprivation in group B (equal to 1) since individual 2 belongs to group B so that the height of point B'' is $0.125+[(1/4)(1/5)] = 0.125+0.050 = 0.175$. The same idea is applied to compute the height of point C''. Starting from B'' we have to add a height which corresponds to the average deprivation in group A since individual 3 belongs to group A and so the height of point C'' is $0.175+[(1/4)(2.5/5)] = 0.175+0.125 = 0.3$. Finally, by adding to the height of point C'' a height corresponding to the average deprivation in group B (individual 4 belongs to group B) we end up with $0.3+[(1/4)(1/5)] = 0.3+0.05 = 0.35$, which is indeed the height of point D. Clearly, the area $OA''B''C''DD'O$ corresponds to half the value of the alternative between groups deprivations while the area $OABCDC''B''A''O$ represents half the value of the within groups deprivation.

It is easy to find out that the area $OA''B''C''DD'O$ is equal to $(0.5*0.25*0.125) + [0.5*0.25*(0.125+0.175)] + [0.5*0.25*(0.175+0.3)] + [0.5*0.25*(0.3+0.35)] = 0.015625+0.0375+0.059375+0.08125 = 0.19375$. Twice this value (0.3875) is hence the value of the alternative between groups deprivation.

This result can also be obtained by applying (22) to the average incomes of the group to which each individual belongs, giving each individual his/her original rank. We then obtain: $(1/16)\{[(7)(2.5/5)]+[(5)(1/5)]+[(3)(2.5/5)]+[(1)(1/5)]\} = (1/80)(17.5+5+7.5+1) = (31/80) = 0.3875$.

The within groups deprivation (the area $OABCDC''B''A''O$) is computed as $[0.5*0.25*(0.2-0.125)] + \{0.5*0.25*[(0.2-0.125)+(0.3-0.175)]\} + \{0.5*0.25*[(0.3-0.175)+(0.35-0.3)]\} + \{0.5*0.25*[(0.35-0.3)]\} = 0.009375+0.025+0.021875+0.00625 = 0.0625$. Twice this area is hence equal to 0.125.

This result may be obtained by applying (20) to the difference for each individual between his/her actual deprivation and the average deprivation of the group to which /she belongs, each individual being assigned again his/her original rank.

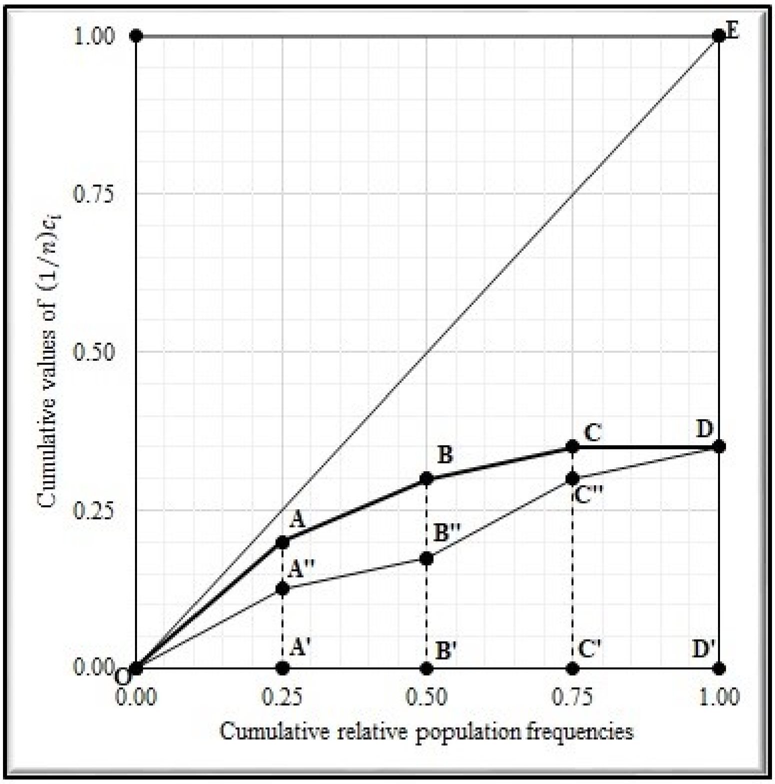

Figure 4: Graphical representation of an alternative decomposition of the MDI

We then get $(1/16)\{[(7)((4-2.5)/5)]+[(5)((2-1)/5)]+[(3)((1-2.5)/5)]+[(1)((0-1)/5)]\} = (1/80)((10.5+5)-(4.5+1)) = 10/80 = 0.125$. The sum of these alternative between and within group's deprivation is hence equal to $0.3875 + 0.125 = 0.5125 = 8.2/16 = MDI$.

Appendix C: The *MDI* as a specific case of the deprivation index of Aaberge et al. (2019). A simple illustration

Let us assume that there are 5 individuals and 10 deprivation domains. Each deprivation has the same weight. Table 3 below indicates how many deprivations each individual has.

The Gini index G_{c_i} of the distribution of the deprivations may then be computed [see expression (4) in Berrebi and Silber, 1983] as $G_{c_i} = [(4/5)(\frac{10-0}{25}) + (2/5)(\frac{7-2}{25})] = (\frac{40+10}{125}) = 0.4$, where 25 in the denominator refers to the total number of deprivations in the population and 5 is the number of individuals. Using (24), we conclude that $MDI = \bar{c}(1 + G_{c_i}) = 5(1 + 0.4) = 7$.

Note that it is also possible to compute the G_{c_i} index using the following formulation of the Gini index (see, Yitzhaki & Schechtman, 2013, p. 15): $G_{c_i} = 2\{\int [1 - F(k)]dk\} - 2\{\int [1 - F(k)]^2 dk\}$.

Using the data of Table 3, we conclude that $\int_0^9 [1 - F(k)]dk = 5$ and that $\int_0^9 [1 - F(k)]^2 dk = 3$. We also conclude that $G_{c_i} = 2(5 - 3)(\frac{1}{10}) = 0.4$.

Since the mean difference Δ_{c_i} of the deprivations is expressed (see Kendall and Stuart, 1969) as $\Delta_{c_i} = 2\bar{c}G_{c_i}$, where $\bar{c}$ is the mean number of deprivations, which is here equal to $(2 + 6 + 7 + 10)/5 = 5$, we conclude that $\Delta_{c_i} = 2 * 5 * 0.4 = 4$.

Aaberge et al. (2019) have suggested using as measure of deprivation in a society an index $D_\Gamma(F)$ defined [see their expression (2.4)] as $D_\Gamma(F) = r - \sum_{k=0}^{r-1} \Gamma(F_k)$, where r refers to the maximum number of deprivation (in our simple illustration $r = 10$). If we take a "union approach", the function Γ has to be convex. A simple convex function would be $\Gamma(F_k) = (F_k)^2$, so that we end up with: $D_\Gamma(F) = r - \sum_{k=0}^{r-1} (F_k)^2$

Using the data of Table 3, we easily find that $\sum_{k=0}^{r-1} (F_k)^2 = \sum_{k=0}^{9} (F_k)^2 = 3$. Since $r = 10$, we conclude that $D_\Gamma(F) = 10 - 3 = 7 = MDI$.

Table 3: A simple numerical illustration

Number k of deprivations	**Number of individuals deprived**	**Relative frequency f_k of deprivations**	**Cumulative relative frequency F_k of deprivations**	$(F_k)^2$	$(1 - F_k)$	$(1 - F_k)^2$	$\int (1 - F_k)$	$\int (1 - F_k)^2$
0	1	0.20	0.20	0.04	0.80	0.64	0.80	0.64
1	0	0.00	0.20	0.04	0.80	0.64	1.60	1.28
2	1	0.20	0.40	0.16	0.60	0.36	2.20	1.64
3	0	0.00	0.40	0.16	0.60	0.36	2.80	2.00
4	0	0.00	0.40	0.16	0.60	0.36	3.40	2.36
5	0	0.00	0.40	0.16	0.60	0.36	4.00	2.72
6	1	0.20	0.60	0.36	0.40	0.16	4.40	2.88
7	1	0.20	0.80	0.64	0.20	0.04	4.60	2.92
8	0	0.00	0.80	0.64	0.20	0.04	4.80	2.96
9	0	0.00	0.80	0.64	0.20	0.04	5.00	3.00
10	1	0.20	1.00	1.00	0.00	0.00		