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Andree Ehlert, Martin Schlather

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Platz der Göttinger Sieben 3 · 37073 Goettingen · Germany
Phone: +49-(0)551-3914066 · Fax: +49-(0)551-3914059

Email: crc-peg@uni-goettingen.de Web: http://www.uni-goettingen.de/crc-peg

Some Results for Extreme Value Processes in Analogy to the Gaussian Spectral Representation

Andree Ehlert* & Martin Schlather†

Georg-August-Universität Göttingen
Goldschmidtstr. 7, D–37077 Göttingen

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Abstract

The extremal coefficient function has been discussed as an analog of the autocovariance function for extreme values. However, as to the behavior of valid extremal coefficient functions little is known apart from their positive definite type. In particular, the reconstruction of valid processes from given extremal coefficient functions has not been considered before. We show, for the one-dimensional case, the equivalence of the set correlation functions and the extremal coefficient functions with finite range on a grid, and study an analogy to Bochner's theorem, namely that any such extremal coefficient function is representable as a convex combination of a finite set of positive definite functions. This allows for the construction of simple max-stable processes complying with a given extremal coefficient function and, in addition, highlights further properties of the latter. We will include an application of this approach and discuss several examples. As to processes with infinite range we will consider a natural extension of the term "long memory" that is well-known in the Gaussian framework to max-stable processes.

Extreme value theory; max-stable process; extremal dependence; extremal coefficient function; set covariance function; set correlation function; homometric; long memory; summability; spectral representation; Bochner's theorem

*ehlert@math.uni-goettingen.de
†schlather@math.uni-goettingen.de

1 Introduction

The study of componentwise maxima for independent copies of stationary processes on $\mathbb{R}^d$ is a natural question arising in extreme value theory. Its relevancy to practice is indicated by numerous applications to extremal phenomena in the environmental or financial context, see e.g. [24, 5, 10]. In theory, the family of limiting processes that emerge from the above setup is fully characterized by the so-called class of max-stable processes. As the latter fails to be of finite parametric nature particular models for max-stable processes have become a major matter of interest. In this regard we may mention the seminal paper by Smith [25], the extensive class of M_4 processes discussed by Smith and Weissman [26], and Schlather [23] for the spatial case. Here, we will focus on certain properties of the dependence structure of max-stable processes. Unlike the Gaussian family, however, where the dependence structure is entirely determined by the corresponding autocovariance function the class of max-stable processes cannot be completely characterized by a similar concept. Still, a suitable summary measure for the dependence structure of such processes is given by the extremal coefficient function, a conditionally negative definite function proposed by Schlather and Tawn [24] that is a special case of the extremogram [6]. For a more generalized point of view we may also refer to the notion of max-zonoids studied in [17]. Similar to the usual autocovariance the extremal coefficient function is a dependence measure for pairwise (temporal or spatial) separations of a process at a given lag $h \in \mathbb{R}^d$. Although it is a rough summary of the dependence structure, i.e. it neither characterizes the multivariate marginals of the process nor the bivariate dependence structure over space or time completely, it has a convenient interpretation that is appropriate to most applications. Moreover, as in the Gaussian case a summability condition on the extremal coefficient function will allow for a corresponding characterization of max-stable processes as having short or long memory. Further, any given extremal coefficient function imposes significant restrictions on the admissible set of underlying max-stable processes. Here, we will exploit in more detail the structure of extremal coefficient functions in order to recover corresponding max-stable processes. Note that throughout we will confine our analysis to the one-dimensional discrete-time case. Based on the well-known fact that the set of extremal coefficient functions is convex [24] we will, in particular, focus on convex decompositions of those functions, i.e. a representation of the latter in terms only of the vertices of their hull, where we will use the term vertex extremal coefficient function in the sense of Sasvári [22], Definition 1.8.1. Up to uniqueness, what will not hold in our analysis here, this is the content of Bochner's theorem, see Theorems 1.8.10 and 1.9.6 in [22], for instance. It will be instructive at this point to have an early look at Fig. 1 below where as an example for a range of $n = 5$ we display the abovementioned vertex extremal coefficient functions. Put differently, all valid extremal coefficient functions on $\mathbb{Z}$ up to range five are given by some convex combination of the functions included in the figure. As a crucial point we will discuss in detail the determination of the set of vertices. Our results also address Matheron's [16] question of characterizing the ensembles of positive definite functions belonging to certain families of marginal distributions. Such characterization, namely that of separable positive definite functions, has been considered by [11].

Our approach will be organized as follows. In Section 2 we will introduce the concept of set correlation functions. Following a discussion of their properties we will restrict to their evaluation on a grid, and determine the vertices of their convex set. We point out that the analysis in Section 2 refrains from any specific aspects of extreme value theory. We will, however, show in Section 3 that the ensembles of set correlation functions and extremal

coefficient functions coincide on a grid. The reason to work with set correlation functions first is that in order to analyze their structure and determine the vertices of their set we may refer directly to well-known concepts from the literature, in particular the problem of homometry [18, 19]. In Section 3 we will then formally refer to the theory of extremes and discuss two particular concepts that are essential to our approach, namely max-stable processes and extremal coefficient functions. In Section 4 we shall introduce a sparse reference class of max-stable processes that is intimately related to the above set of vertices. The class of processes depends on a weight vector that may be chosen such as to reproduce any valid extremal coefficient function. The reconstruction of max-stable example processes from given extremal coefficient functions is then essentially reduced to the determination of suitable weights. An example of the latter in addition to some related applications will be discussed in Section 5. Finally, the usefulness of partial knowledge of the extremal coefficient function for assertions on the range of the underlying process will be considered in Section 6.
Throughout, following standard conventions we will write $S + q = \{x + q : x \in S\}$, and accordingly $aS = \{ax : x \in S\}$, for a set $S \subseteq \mathbb{R}$, and $q, a \in \mathbb{R}$. We will denote the indicator function of a set $S \subseteq \mathbb{R}$ by $\mathbf{1}(x \in S)$. Further, we will assume all operations that involve vectors to apply componentwise, and denote by "$\subset$" a proper inclusion whereas "$\subseteq$" does not preclude equality. For $x \in \mathbb{R}$ let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

2 Set correlation functions and basic notions

In this section we will concentrate exclusively on set correlation functions, a concept shown in Section 3 to be equivalent to the extremal coefficient functions on a grid. Note that this section is self-contained and independent of the concepts used in extreme value theory. In our approach we will prove stepwise an analog of Bochner's theorem in the sense of Sasvári's [22] Theorem 1.9.6 for the ensemble $\mathcal{F}^*_{n,\mathbb{Z}}$ of set correlation functions with finite range $n \in \mathbb{N}$ that are evaluated on $\mathbb{Z}$. That is, we will show that $\mathcal{F}^*_{n,\mathbb{Z}}$ is a convex set, and we will determine its vertices, i.e. the extremal positive definite functions of $\mathcal{F}^*_{n,\mathbb{Z}}$, see Lemma 1 and Theorem 1 below. To begin with, it will be instructive to incorporate the relevant concepts successively into the well-known framework of general covariograms. To this end, for an integrable and square integrable function $w(x)$ in $\mathbb{R}$ we define the covariogram by the convolution product

$$f(h) = \int w(x)w(x+h)dx, \quad h \in \mathbb{R}. \tag{1}$$

Note that two fundamental properties of the covariogram, namely symmetry and positive definiteness, are immediate from (1), and will be crucial in the following. As an important special case of (1) we will consider next the length of the intersection of a set with its translation. More precisely, for $w(x) = \mathbf{1}(x \in S)$, $S \in \sigma_\infty$, let

$$f_S(h) = \int \mathbf{1}(x \in S)\mathbf{1}(x \in (S-h))dx = |S \cap (S-h)|, \quad h \in \mathbb{R}, \tag{2}$$

denote the set covariance function of S, also termed geometric covariogram [15], where σ_∞ stands for the ensemble of all Borel sets $S \subseteq \mathbb{R}$ with $0 < |S| < \infty$. For later reference we introduce $\sigma_n \subseteq \sigma_\infty$ in order to represent accordingly all Borel sets $S \subseteq [q, n+q)$ for some $q \in \mathbb{R}$. The number $n \in \mathbb{N}$ will later be referred to as finite range. For convenience, in the following we shall without loss of generality consider the set correlation functions $f^*_S(h) = f_S(h)/f_S(0)$

for all $h \in \mathbb{R}$, $S \in \sigma_\infty$. To provide some preliminary insight into the behavior of f_S^* note that by (2) we have in particular that $f_S^*(0) = 1$, $\int f_S^*(h)dh = |S|$, and that $f_S^*(h)$ is not differentiable at the origin [4]. As a further restriction of (1) and (2) we shall henceforth confine our analysis to the evaluation of f_S^* on a subset $Q \subseteq \mathbb{R}$, i.e. we consider $f_S^*(h)$, $h \in Q$, and put $\mathcal{F}_{n,Q}^* = \{f_S^* \in \mathbb{R}^Q : S \in \sigma_n\}$ for any $n \in \mathbb{N} \cup \{\infty\}$. Note that $f_S^* \in \mathcal{F}_{\infty,Q}^*$ might be in $\mathcal{F}_{n,Q}^*$ for some $n \in \mathbb{N}$ although S is unbounded. The following elementary lemma provides a fundamental background for the rest of our analysis.

Lemma 1. *For all $n \in \mathbb{N} \cup \{\infty\}$ and all $p \in \mathbb{N}$ the set $\mathcal{F}_{n,p^{-1}\mathbb{Z}}^*$ is convex.*

Proof. Let $f_{S_1}^*, f_{S_2}^* \in \mathcal{F}_{n,p^{-1}\mathbb{Z}}^*$, $n \in \mathbb{N} \cup \{\infty\}$. Consider first the case $n \in \mathbb{N}$. Without loss of generality we may assume that $S_i \subseteq [0, n)$, $i = 1, 2$. For $\lambda \in [0, 1]$ put

$$\begin{aligned} S_3 \quad = \quad & \bigcup_{i \in \mathbb{Z}} \left[\left(\left[0, \frac{\lambda}{p}\right) \cap \lambda \left(S_1 - \frac{i-1}{p} \right) \right) \right. \\ & \left. \cup \left(\left(\left[0, 1 - \frac{\lambda}{p}\right) \cap (1-\lambda) \left(S_2 - \frac{i-1}{p} \right) \right) + \frac{\lambda}{p} \right) + \frac{i-1}{p} \right]. \end{aligned}$$

Now, we have that $S_3 \in \sigma_n$, and $f_{S_3}^*(h) = \lambda f_{S_1}^*(h) + (1-\lambda) f_{S_2}^*(h)$, $h \in \mathbb{Z}/p$, holds by (2). If $n = \infty$, the assertion follows for $S_i \subseteq \mathbb{R}$, $i = 1, 2$. □

Next, by $V(\mathcal{F}_{n,\mathbb{Z}}^*)$ we will denote the unknown set of vertices representing the convex hull of $\mathcal{F}_{n,\mathbb{Z}}^*$. It will be a consequence of Proposition 1 below that $V(\mathcal{F}_{n,\mathbb{Z}}^*)$ is contained in a natural superset with finite cardinality for any $n \in \mathbb{N}$, i.e. $|V(\mathcal{F}_{n,\mathbb{Z}}^*)| \leq 2^n$. The superset will be determined by the set of all 2^n binary vectors that itself entails substantial redundancies to be discussed below. We will introduce simple set correlation functions $f_{U_b}^*$ for $U_b = \bigcup_{j \in I_b} [j-1, j)$, where I_b is the set of indices corresponding to ones in $b = (b_1, \ldots, b_n) \in \mathcal{B}_n = \{0, 1\}^n$ (e.g. $I_b = \{1, 3, 4\}$ for $b = (1, 0, 1, 1)$). For the restriction of $f_{U_b}^*$, $b \in \mathcal{B}_n$, to $\mathbb{Z}$ we shall for simplicity introduce the notation $f_{I_b}^*$, and put $\mathcal{H}_{n,\mathbb{Z}}^* = \left\{ f_{I_b}^* \in \mathbb{R}^{\mathbb{Z}}, b \in \mathcal{B}_n \right\}$. For later reference, note that by (2), in particular,

$$f_{I_b}^*(h) = \sum_{k \in \mathbb{Z}} \min\{b_k, b_{k+h}\} |I_b|^{-1} = \sum_{k \in \mathbb{Z}} b_k b_{k+h} |I_b|^{-1}, \quad h \in \mathbb{Z},\, b \in \mathcal{B}_n, \tag{3}$$

where $b_k = 0$ for $k \in \mathbb{Z} \setminus \{1, \ldots, n\}$.

Proposition 1. *For all $n \in \mathbb{N}$ we have that $V(\mathcal{F}_{n,\mathbb{Z}}^*) \subseteq \mathcal{H}_{n,\mathbb{Z}}^*$. Further, $V(\mathcal{F}_{\infty,\mathbb{Z}}^*) \subseteq \bigcup_{n=1}^{\infty} \mathcal{H}_{n,\mathbb{Z}}^*$.*

Proof. In order to show the first assertion let $n \in \mathbb{N}$ and $S \in \sigma_n$. Without loss of generality we may assume that $S \subseteq [0, n)$. We will show that

$$f_S^*(h) = \sum_{b \in \mathcal{B}_n} f_{I_b}^*(h) \mu_b, \quad h \in \mathbb{Z},$$

where $0 \leq \mu_b \leq 1$, $b \in \mathcal{B}_n$, $\sum_{b \in \mathcal{B}_n} \mu_b = 1$. To this end, for all $b \in \mathcal{B}_n$ let

$$\delta_b = [0, 1) \cap \bigcap_{i \in I_b} (S + 1 - i) \tag{4}$$

and put

$$\Delta_b = \delta_b \cap \bigcap_{a \in \mathcal{B}_n : I_b \subset I_a} \delta_a^c \subseteq [0,1). \tag{5}$$

Now, we find that

$$\Delta_a \cap \Delta_b = \delta_a \cap \delta_b \cap \Bigg(\bigcup_{\substack{\omega \in \mathcal{B}_n : I_a \subset I_\omega \\ \text{or } I_b \subset I_\omega}} \delta_\omega \Bigg)^c = \emptyset \quad \text{for all } a, b, \in \mathcal{B}_n \text{ with } a \neq b. \tag{6}$$

Here, the last equality follows from the fact that by (4) we have

$$\delta_a \cap \delta_b \subseteq \bigcup_{\substack{\omega \in \mathcal{B}_n : I_a \subset I_\omega \\ \text{or } I_b \subset I_\omega}} \delta_\omega, \quad a, b, \in \mathcal{B}_n,\ a \neq b.$$

Let $S_b = \dot\bigcup_{i \in I_b} (\Delta_b + i - 1)$, $b \in \mathcal{B}_n$, where the union is disjoint by (5). By (5) and (6) we get in particular that

$$S_a \cap (S_b + h) = \emptyset \quad \text{for all } h \in \mathbb{Z}, \text{ and all } a, b, \in \mathcal{B}_n \text{ with } a \neq b. \tag{7}$$

Further, (4) and (5) yield for all $b \in \mathcal{B}_n$ that $S_b = \dot\bigcup_{i \in I_b} (\Delta_b + i - 1) \subseteq S$, and hence

$$\dot\bigcup_{b \in \mathcal{B}_n} S_b \subseteq S. \tag{8}$$

Next, note that $x \in S$ by (4) implies that $x \in \delta_b + i - 1$ for some $i \in \{1, \ldots, n\}$ and $b \in \mathcal{B}_n$ with $I_b = \{i\}$. By (5) we get further that $x \in \Delta_a + i - 1$ for some $a \in \mathcal{B}_n$ with $I_b \subseteq I_a$. Altogether we now find that $x \in S$ implies

$$x \in \dot\bigcup_{b \in \mathcal{B}_n} (\Delta_b + i - 1) \subseteq \dot\bigcup_{b \in \mathcal{B}_n} \dot\bigcup_{i \in I_b} (\Delta_b + i - 1) = \dot\bigcup_{b \in \mathcal{B}_n} S_b,$$

and hence $S = \dot\bigcup_{b \in \mathcal{B}_n} S_b$ by (8). Then, from (7) and (2) we get for $\mu_b = |S_b|/|S| = |\Delta_b||I_b|/|S|$, $b \in \mathcal{B}_n$, that

$$f_S^*(h) = \sum_{b \in \mathcal{B}_n} f_{S_b}^*(h) \mu_b = \sum_{b \in \mathcal{B}_n} f_{I_b}^*(h) \mu_b$$

where the second equality holds by definition of S_b and $f_{I_b}^*$. We finally consider the second assertion. For any $S \in \sigma_\infty$ let

$$L_n = \bigcup_{z \in \mathbb{Z}} \Bigg(z + \bigcup_{i \in \mathbb{Z} \setminus \{-n, \ldots, n\}} ((S - i) \cap [0,1]) \Bigg), \quad n \in \mathbb{N},$$

and $S = (S \cap L_n) \cup (S \cap L_n^c)$. Then,

$$f_S^* = |S \cap L_n^c||S|^{-1} f_{S \cap L_n^c}^* + |S \cap L_n||S|^{-1} f_{S \cap L_n}^* \in \mathcal{F}_{\infty,\mathbb{Z}}^*$$

and $f_{S \cap L_n^c}^* \in \mathcal{F}_{2n+1,\mathbb{Z}}^*$. Now, for $n \to \infty$ we have that $|S \cap L_n^c||S|^{-1} \to 1$, and the second summand tends to 0. $\square$

Next, via the introduction of suitable equivalence relations we will successively discard redundancies within $\mathcal{B}_n$ and finally determine a set $\mathcal{C}_n \subseteq \mathcal{B}_n$ with $V(\mathcal{H}^*_{n,\mathbb{Z}}) = \left\{ f^*_{I_b} \in \mathbb{R}^{\mathbb{Z}}, b \in \mathcal{C}_n \right\}$. In particular, we will demonstrate that the immediate idea of congruence for any two sets I_a and I_b, $a, b \in \mathcal{B}_n$, is a sufficient condition for $f^*_{I_a} = f^*_{I_b}$ only whereas the concept of homometry to be discussed below is necessary and sufficient. Still, we will also study the former equivalence relation in more detail as the number of noncongruent and homometric vectors $a, b \in \mathcal{B}_n$ will turn out to be relatively small, cf. Proposition 2 and Tab. 1. To formalize the notion of congruence first define reflections $r_u : \{0,1\}^n \to \{0,1\}^n$, $u \in \{0,1\}$, $r_1((x_1, \dots, x_n)) = (x_n, \dots, x_1)$, $r_0 = id$, and translations $s_t : \{0,1\}^n \to \{0,1\}^n$, $t \in \mathbb{Z}$,

$$s_t((x_1, \dots, x_n)) = \begin{cases} (0, \dots, 0, x_1, \dots, x_{n-t}) & \text{if } x_{n-t+1}, \dots, x_n = 0 \text{ and } t \geq 0, \\ (x_{-t+1}, \dots, x_n, 0, \dots, 0) & \text{if } x_1, \dots, x_{-t} = 0 \text{ and } t \leq -1, \\ (x_1, \dots, x_n) & \text{else.} \end{cases}$$

Now, for all $a, b \in \mathcal{B}_n$ we will define congruence by the equivalance relation $a \sim_c b$, $a = s_t \circ r_z(b)$ for some $(t, z) \in \{-n+1, \dots, n-1\} \times \{0,1\}$. We denote the quotient set of $\mathcal{B}_n$ with respect to $\sim_c$ by $\mathcal{B}_n/{\sim_c}$ and state the following result for $|\mathcal{B}_n/{\sim_c}|$, i.e. the number of non-congruent patterns in $\mathcal{B}_n$.

Proposition 2. *We have that*

$$|\mathcal{B}_n/{\sim_c}| = 2^{n-2} + 2^{\lfloor (n-2)/2 \rfloor} + 2^{\lfloor (n-1)/2 \rfloor} - 1, \quad n \in \mathbb{N}. \tag{9}$$

In particular, we have $|\mathcal{B}_n/{\sim_c}| \sim 2^{n-2}$.

Proof. Let $\mathcal{B}_{n,1} = \{b \in \mathcal{B}_n : b_1 = 1\} \subseteq \mathcal{B}_n$ where applying the translation defined above we have that $b = s_t(a)$ for all $b \in \mathcal{B}_n$ and some $(t, a) \in \{0, \dots, n-1\} \times \mathcal{B}_{n,1}$. Hence, by definition of the equivalence relation $\sim_c$ we find that

$$|\mathcal{B}_n/{\sim_c}| = |\mathcal{B}_{n,1}/{\sim_c}|. \tag{10}$$

Next, consider the partition $\mathcal{B}_{n,1,N} \cup \mathcal{B}_{n,1,E}$ of $\mathcal{B}_{n,1}$ where $\mathcal{B}_{n,1,N} = \{b \in \mathcal{B}_{n,1} : b_n = 0\}$ and $\mathcal{B}_{n,1,E} = \mathcal{B}_{n,1} \setminus B_{n,1,N}$. We obviously get that $a \not\sim_c b$ for any $a \in \mathcal{B}_{n,1,N}$ and any $b \in \mathcal{B}_{n,1,E}$ such that

$$|\mathcal{B}_{n,1}/{\sim_c}| = |\mathcal{B}_{n,1,N}/{\sim_c}| + |\mathcal{B}_{n,1,E}/{\sim_c}|. \tag{11}$$

Note that by definition of $\mathcal{B}_{n,1}$ and $\mathcal{B}_{n,1,N}$ we have that $b \in \mathcal{B}_{n-1,1}$ if and only if $(b, 0) \in \mathcal{B}_{n,1,N}$ such that, in particular, $|\mathcal{B}_{n-1,1}/{\sim_c}| = |\mathcal{B}_{n,1,N}/{\sim_c}|$. Applying the latter equality successively to (11) we find with (10) that

$$|\mathcal{B}_n/{\sim_c}| = \sum_{j=1}^{n} |\mathcal{B}_{j,1,E}/{\sim_c}|. \tag{12}$$

For $S_n = \{b \in \mathcal{B}_{n,1,E} : b_k = b_{n-k+1}, k = 1, \dots, n\}$, i.e. the set of all symmetric vectors $b \in \mathcal{B}_{n,1,E}$, we now consider the partition

$$\mathcal{B}_{n,1,E} = A_n \cup S_n \tag{13}$$

where $A_n = B_{n,1,E} \setminus S_n$. It is immediate that S_n can be identified with its quotient set with respect to $\sim_c$, i.e. $S_n/{\sim_c} = \bigcup_{S \in S_n} \{S\}$. Moreover, with respect to the set $A_n \subseteq \mathcal{B}_{n,1,E}$ of asymmetric vectors for all $a \in A_n$ we have that $r_1(a) = b$ for some $b \in A_n$, $b \neq a$. Note that $s_t(a) = a$ for all $(t, a) \in \mathbb{Z} \times \mathcal{B}_{n,1,E}$, and hence we get that $|A_n/\sim_c| = \frac{1}{2}|A_n|$. Further, the definition of S_n yields that $a \not\sim_c b$ for any $a \in S_n$ and any $b \in A_n$ such that

$$|\mathcal{B}_{n,1,E}/\sim_c| = |A_n/\sim_c| + |S_n/\sim_c| = \frac{1}{2}|A_n| + |S_n| = \frac{1}{2}|\mathcal{B}_{n,1,E}| + \frac{1}{2}|S_n| \tag{14}$$

where the second equality follows from the above results and the third equality holds by (13). Note that $\mathcal{B}_{n,1,E}$ is the ensemble of all $b \in \mathcal{B}_n$ with $b_1 = b_n = 1$ and cardinality

$$|\mathcal{B}_{n,1,E}| = \sum_{m=0}^{n-2} \binom{n-2}{m}. \tag{15}$$

For the number of symmtric sequences $|S_n|$ we find by case differentiation that, for $n \geq 3$,

$$|S_n| = \sum_{m=0}^{n-2} \begin{cases} \binom{\frac{1}{2}(n-2)}{\frac{1}{2}m} & \text{if } m, n \text{ even,} \\ \binom{\frac{1}{2}(n-3)}{\frac{1}{2}m} & \text{if } n \text{ odd and } m \text{ even,} \\ \binom{\frac{1}{2}(n-3)}{\frac{1}{2}(m-1)} & \text{if } m, n \text{ odd,} \\ 0 & \text{else.} \end{cases} \tag{16}$$

By case differentiation upon (9) we get that, for $n \geq 3$,

$$\begin{aligned} |\mathcal{B}_n/\sim_c| &= 4 + \frac{1}{2}\sum_{j=2}^{n-2}\sum_{m=0}^{j}\binom{j}{m} + \frac{1}{2}\sum_{j=1}^{\lfloor(n-2)/2\rfloor}\sum_{m=0}^{j}\binom{j}{m} + \sum_{j=1}^{\lfloor(n-3)/2\rfloor}\sum_{m=0}^{j}\binom{j}{m} \\ &= 2^{n-2} + 2^{\lfloor(n-2)/2\rfloor} + 2^{\lfloor(n-1)/2\rfloor} - 1. \end{aligned}$$

It is readily seen that the r.h.s. also holds for $n = 1$ and $n = 2$. □

Note from Proposition 2 that the correction for congruence asymptotically reduces the number of relevant binary vectors by three quarters. Next, in order to motivate the notion of homometry we shall consider an alternative interpretation of the set correlation that focusses on the mutual differences between the elements in I_b, i.e.

$$f^*_{I_b}(h) = |\{(x, y) \in I_b^2 : x + h = y\}||I_b|^{-1}, \quad h \in \mathbb{Z}. \tag{17}$$

The concept of homometry, also known as turnpike or partial digest problem, is typically specified by equations similar to (17). In particular, given all distances between points on the line is it possible to retrieve the corresponding sets I_b, $b \in \mathcal{B}_n$, up to congruence? Put differently, if any is there a unique class $[b] \in \mathcal{B}_n/\sim_c$, $b \in \mathcal{B}_n$, identified by a given set correlation function $f^* \in \mathcal{H}^*_{n,\mathbb{Z}}$? The answer goes back at least as far as [18, 19] in the context of the analysis of diffraction patterns in crystallography where the set covariance is related to so-called multisets and where it is also well-known that $|\mathcal{B}_n/\sim_c| > |\mathcal{H}^*_{n,\mathbb{Z}}|$, $n \geq 12$, cf. Table 1. In line with the above discussion two patterns $a, b \in \mathcal{B}_n$ are called homometric if, $a \sim_h b$, $f^*_{I_a} = f^*_{I_b}$, cf. [19]. Denote by $[a] = \{b \in \mathcal{B}_n : b \sim_h a\}$ the equivalence class of a, i.e. we identify the equivalence class $[a]$ with the corresponding function $f^*_{I_a}$ and put $f^*_{[a]} = f^*_{I_a}$, $a \in \mathcal{B}_n$. Let

$$\mathcal{B}_n/\sim_h = \{[b] : b \in \mathcal{B}_n\}.$$

Emphasizing computational complexity the problem has been discussed more recently in [14]. In particular, the determination of some $b \in [a]$ from $f^*_{[a]}$ appears to be NP-complex in general. For later reference we may define $|[b]| := |I_a|$ for some $a \in [b]$, $b \in \mathcal{B}_n$, by the following lemma.

Lemma 2. *From $a \in [b]$ for some $b \in \mathcal{B}_n$ it follows that $|I_a| = |I_b|$.*

Proof. By definition we have that $a \in [b]$, $b \in \mathcal{B}_n$, is equivalent to $f^*_{I_a} = f^*_{I_b}$ such that, in particular, $\max I_a - \min I_a = \max I_b - \min I_b =: r$. The latter yields by (3) that $f_{I_a}(r) = f_{I_b}(r) = 1$. Now, by definition we get $|I_a| = f_{I_a}(r)/f^*_{I_a}(r) = f_{I_b}(r)/f^*_{I_b}(r) = |I_b|$. $\square$

By the above discussion we may now restrict to any representative of $\mathcal{B}_n/\sim_h$ as candidate vectors generating $V(\mathcal{H}^*_{n,\mathbb{Z}})$. For

$$\mathcal{C}_n = \left\{[a] \in \mathcal{B}_n/\sim_h : f^*_{[a]} \neq \sum_{[b]\in\mathcal{B}_n/\sim_h\setminus\{[a]\}} f^*_{[b]}\mu_{[b]}, \text{ for all } \mu_{[b]} \in [0,1]\right\} \subseteq \mathcal{B}_n/\sim_h \tag{18}$$

we get that

$$\left\{f^*_{I_b} \in \mathbb{R}^{\mathbb{Z}} : b \in \mathcal{C}_n\right\} = V(\mathcal{H}^*_{n,\mathbb{Z}}) = V(\mathcal{F}^*_{n,\mathbb{Z}}) \tag{19}$$

where the second equality follows from Proposition 1 and the fact that $\mathcal{H}^*_{n,\mathbb{Z}} \subseteq \mathcal{F}^*_{n,\mathbb{Z}}$. Note that beyond the idea of homometry we are not aware of a suitable concept that yields the set $\mathcal{C}_n$ directly from $\mathcal{B}_n$. The following theorem gives an analog of Bochner's theorem (cf. e.g. Theorems 1.9.6 in [22]) for set correlation functions on $\mathbb{R}$ that are evaluated on $\mathbb{Z}$ only.

Theorem 1. *For all $S \in \sigma_n$, $n \in \mathbb{N} \cup \{\infty\}$, there is $\mathcal{X} \subseteq \mathcal{C}_n$ such that*

$$f^*_S(h) = \sum_{[b]\in\mathcal{X}} f^*_{[b]}(h)\mu_{[b]}, \quad h \in \mathbb{Z}, \tag{20}$$

where $0 < \mu_{[b]} \leq 1$, $[b] \in \mathcal{X}$, $\sum_{[b]\in\mathcal{X}} \mu_{[b]} = 1$. Reversely, given the r.h.s. of (20) a set $S \in \sigma_n$ exists such that (20) holds. In particular, $|\mathcal{X}| \leq n$, $n \in \mathbb{N}$.

Proof. The proof of Proposition 1 yields that for all $S \in \sigma_n$, $\bar{\mu}_b = |S_b|/|S|$ and $h \in \mathbb{Z}$ we have

$$f^*_S(h) = \sum_{b\in\mathcal{B}_n} f^*_{S_b}(h)\bar{\mu}_b = \sum_{b\in\mathcal{B}_n} f^*_{I_b}(h)\bar{\mu}_b = \sum_{[b]\in\mathcal{C}_n} f^*_{[b]}(h)\hat{\mu}_{[b]} = \sum_{[b]\in\mathcal{X}\subseteq\mathcal{C}_n} f^*_{[b]}(h)\mu_{[b]},$$

where the third equality follows by (19). The existence of $\mathcal{X} \subseteq \mathcal{C}_n$ with $|\mathcal{X}| \leq n$, is a consequence of Carathéodory's theorem [3] and (19). Finally, note that

$$\sum_{[b]\in\mathcal{C}_n} \hat{\mu}_{[b]} = \sum_{[b]\in\mathcal{X}} \mu_{[b]} = \sum_{b\in\mathcal{B}_n} \bar{\mu}_b = 1$$

where the weights $\hat{\mu}_{[b]}$, $[b] \in \mathcal{C}_n$, and $\mu_{[b]}$, $[b] \in \mathcal{X}$, are not unique in general. $\square$

The fact that in general $\mathcal{B}_n/\sim_h$ may include interior points of $\mathcal{C}_n$ is referred to in Table 1 where it is shown that $|\mathcal{B}_n/\sim_h| > |\mathcal{C}_n|$ for $n \geq 5$, cf. also Section 5.3. Note that the results for $|\mathcal{B}_n/\sim_h|$ and $|\mathcal{C}_n|$ in Table 1 have been obtained by simulation, cf. [2]. Related questions have also been studied by [12] and [20].

n	$\vert\mathcal{B}_n\vert = 2^n$	$\vert\mathcal{B}_n/{\sim_c}\vert$	$\vert\mathcal{B}_n/{\sim_c}\vert - \vert\mathcal{B}_n/{\sim_h}\vert$	$\vert\mathcal{B}_n/{\sim_h}\vert - \vert\mathcal{C}_n\vert$
4	16	7	0	0
5	32	13	0	1
6	64	23	0	2
7	128	43	0	2
8	256	79	0	4
9	512	151	0	7
10	1024	287	0	19
11	2048	559	0	36
12	4096	1087	2	73
13	8192	2143	8	131
14	16384	4223	20	259
15	32768	8383	36	523
16	65536	16639	73	958
17	131072	33151	128	1762
18	262144	66047	234	3379
19	524288	131839	394	–
20	1048576	263167	682	–

Table 1: Number of equivalence classes with respect to congruence, cf. Proposition 2, and homometry where $|\mathcal{H}^*_{n,\mathbb{Z}}| = |\mathcal{B}_n/{\sim_h}|$. We also state the number of homometric equivalence classes in the interior of $\mathcal{C}_n$, i.e. a number $|\mathcal{B}_n/{\sim_h}| - |\mathcal{C}_n|$ of set correlation functions $f^*_{[b]}$, $b \in \mathcal{B}_n/{\sim_h}$, are convex combinations of some $f^*_{[a]}$, $[a] \in \mathcal{C}_n$, $a \neq b$, cf. Eq. (18). The latter result has been obtained by a search algorithm and gives a lower bound. Because of computational limitations we do not report the results for $n \geq 19$.

3 Relations between extremal coefficient and set correlation

A stochastic process $X = (X_t, t \in \mathbb{Z})$ with standard Fréchet margins is called max-stable if its finite dimensional distributions satisfy

$$P(X_1 \leq x_1, \ldots, X_k \leq x_k) = \exp\left(- \int_0^1 \bigvee_{i=1}^k \frac{\tilde{\gamma}_i(s)}{x_i} ds \right) \tag{21}$$

for any $k \geq 1$ and all $z_i \geq 0$, $i = 1, \ldots, k$. Here, the so-called spectral functions $\tilde{\gamma}_i : [0,1] \to \mathbb{R}_+$ are such that $\int_0^1 \tilde{\gamma}_i(s) ds = 1$ for all i. According to [8] the max-stable process X can be represented as

$$X_t = \max_{i \in \mathbb{N}} U_i \gamma_{S_i}(t), \quad t \in \mathbb{Z},$$

where $\{(U_i, S_i)\}_{i=1}^\infty$ is a Poisson point process with intensity $u^{-2}\mathbf{1}(u > 0)du \times \mathbf{1}(s \in [0,1])ds$, and $\gamma_i : \mathbb{R} \to \mathbb{R}_+$, $\int \gamma_i(s) ds = 1$ for all i. By [27] a dissipative stationary max-stable process Y has the representation

$$Y_t = \max_{i \in \mathbb{N}} U_i \gamma_{t - z_i}(S_i), \quad t \in \mathbb{Z},$$

where $\{(U_i, S_i, z_i)\}_{i=1}^\infty$ is a Poisson point process on $[0, \infty) \times \mathcal{S} \times \mathbb{Z}$ with intensity measure $u^{-2}\mathbf{1}(u > 0)du \times dS \times 1$. Without loss of generality we may again assume that $\mathcal{S} = [0,1]$ and $dS = \mathbf{1}(s \in [0,1])ds$. For $g_0(t) = \gamma_{\lfloor t \rfloor}(t - \lfloor t \rfloor)$, $t \in \mathbb{R}$, we have that

$$Y_t = \max_{i \in \mathbb{N}} U_i g_0(t - z_i), \quad t \in \mathbb{Z}, \tag{22}$$

where $\{(U_i, z_i)\}_{i=1}^{\infty}$ is a Poisson point process on $[0, \infty) \times \mathbb{R}$ with intensity measure $u^{-2}\mathbf{1}(u > 0)du \times dz$. Note that in (22) the spectral function g_0 completely charaterizes the dependence structure of the max-stable process Y on $\mathbb{Z}$. Further, the range of Y is given by

$$r_Y = \inf\{m \in \mathbb{N} : |\text{supp}(g_0) \cap (\text{supp}(g_0) + t)| = 0 \text{ for all } |t| \geq m, \, t \in \mathbb{Z}\},$$

i.e. $(Y_1, \ldots, Y_k)$ and $(Y_{k+q}, \ldots, Y_{k+q+l})$ are independent for all $q \geq r_Y$, $k, l \in \mathbb{N}$. As a summary measure reflecting the temporal (spatial) dependence structure of Y the metric

$$d_{g_0}(h) = \int |g_0(s) - g_0(s+h)| ds, \quad h \in \mathbb{Z}, \tag{23}$$

has been proposed in [7]. Following its standard usage in the literature we shall not directly refer to d_{g_0} but define the equivalent extremal coefficient function [24] as a transformation of (23) given by

$$\phi_{g_0}(h) = \frac{d_{g_0}(h) + 2}{2} = \int \max\{g_0(s), g_0(s+h)\} ds, \quad h \in \mathbb{Z}. \tag{24}$$

Note that a more intuitive interpretation of $\phi_{g_0}(h)$ using (21) is given by

$$P(Y_{g_0,0} \leq y, Y_{g_0,h} \leq y) = P(Y_{g_0,0} \leq y)^{\phi_{g_0}(h)}, \quad y > 0, \tag{25}$$

or, alternatively,

$$\phi_{g_0}(h) = 2 - \lim_{y \to \infty} P(Y_{g_0,h} > y \mid Y_{g_0,0} > y). \tag{26}$$

Both representations particularly emphasize the relevancy to practice of the extremal coefficient function, cf. [10]. Especially (26) provides a convenient interpretation in terms of the conditional probability of an extreme event to follow a preceding extreme event at lag h. Note that $\phi_Y(h) = 2$, $h \in \mathbb{Z}$, by (25) is equivalent to independence of Y_t and Y_{t+h} for all $t \in \mathbb{Z}$. For the ensemble of extremal coefficient functions we will consider a summability condition, and put $\Phi_{\infty,\mathbb{Z}} = \left\{\phi \in [0,2]^{\mathbb{Z}} : \sum_{h \in \mathbb{Z}} (2 - \phi(h)) < \infty\right\}$. We will denote by $\Phi_{n,\mathbb{Z}}$ the restriction of $\Phi_{\infty,\mathbb{Z}}$ to those underlying processes X with finite range $r_X \leq n$. The above classification of extremal coefficient functions motivates the following analogy to the term "long memory" [1] that usually refers to the non-summabilty of the autocovariance function in the Gaussian framework. We will propose an analogous notion for max-stable processes.

Definition 1. *A second order weakly stationary random field on $\mathbb{Z}$ with covariance function ρ has a long memory [1] if $\sum_{h \in \mathbb{Z}} |\rho(h)| = \infty$. A stationary random field Y on $\mathbb{Z}$ with existing extremal coefficient function ϕ has a long memory if $\phi_Y \notin \Phi_{\infty,\mathbb{Z}}$, i.e. the correlation function of the random field $\mathbf{1}(Y > n)$ is not absolutely summable in the limit as $n \to \infty$.*

Proposition 3. *Any stationary max-stable process Y on $\mathbb{Z}$ with standard Fréchet margins and summable function $2 - \phi$ is dissipative.*

Proof. Let $Y_t = {}^{e}\!\!\int_R f_t(u) M_\alpha(du)$, $t \in \mathbb{Z}$, for some measure space R and some random measure M_α, see [27] for details. Then R can be uniquely decomposed into a dissipative part and a conservative part C [27, Theorem 6.2], where $\sum_{t \in \mathbb{Z}} f_t(u) = \infty$, $u \in C$. Assume that $\mu(C) > 0$

where μ is the control measure. By stationarity, $\mu(C_0) > 0$ for $C_0 = \{u \in C : f_0(u) > 0\}$, and we get

$$\sum_t \lim_{x\to\infty} \frac{P(Y_0 \geq x, Y_t \geq x)}{1-\exp(-1/x)} \geq \sum_t \lim_{x\to\infty} \frac{1-\exp\left(-x^{-1}\int_{C_0}\min\{f_t(u), f_0(u)\}\mu(du)\right)}{1-\exp(-1/x)}$$
$$= \int_{C_0}\sum_t \min\{f_t(u), f_0(u)\}\mu(du) = \infty$$

from the fact that $\sum_t f_t(u) = \infty$ implies $\sum_t \min\{f_t(u), f_0(u)\} = \infty$ for all $u \in C_0$. $\square$

The following theorem is essential to the integration of the results discussed in Section 2 into the extreme value context. It characterizes every summable function $2-\phi$ for max-stable processes on $\mathbb{Z}$ as a special set correlation function.

Theorem 2. *For all $n \in \mathbb{N}\cup\{\infty\}$ we have $\mathcal{F}^*_{n,\mathbb{Z}} = \{2-\phi : \phi \in \Phi_{n,\mathbb{Z}}\}$.*

Proof. Let $\xi \in \mathcal{F}^*_{n,\mathbb{Z}}$, $n \in \mathbb{N}\cup\{\infty\}$. Then, there is $S \in \sigma_n$ such that $f^*_S(h) = \xi(h)$, $h \in \mathbb{Z}$. Further, for $g_0(x) = \mathbf{1}(x \in S)|S|^{-1}$ we have $\phi_{g_0} = 2-\xi \in \Phi_{n,\mathbb{Z}}$ by (24) and (2). The reverse direction is a direct consequence of Corollary 3 in [9], and Proposition 3. $\square$

Now, Theorem 2 yields in particular that a discrete-time max-stable random field has a long memory if and only if its extremal coefficient function cannot be represented by a set correlation function. Note that Definition 1 also characterizes certain dissipative processes as having a long memory. Our point of view therefore differs from the interpretation in [21] where the definition for short memory phenomena coincides with a process being purely dissipative. Consider e.g. a dissipative process as in (22) with spectral function $g_0(s) = s^{-2}\mathbf{1}(s \geq 1)$ that has a long memory according to Definition 1. With respect to Theorem 2 note also that we have discussed three equivalent concepts representing the extremal coefficient function on a grid, namely ϕ, d and f, cf. (24), (23) and (2). We will henceforth mainly be concerned with two questions related to the above setup. Namely, in what way is the class of extremal coefficient functions restricted by the right-hand side of (24), and how can processes of the form given in (22) be reconstructed for given extremal coefficient functions? To this end, from now on we will focus on so-called M_3 processes, also termed mixed moving maxima [28]. More precisely, the processes are discrete versions of (22) where

$$M_t = \max_{j=1}^{J}\max_{k\in\mathbb{Z}} a_{jk}U_{j,t-k}, \quad t \in \mathbb{Z}, \tag{27}$$

for some $J \in \mathbb{N}$ and a sequence $\{U_{ji}, j \in \{1,\dots,J\}, i \in \mathbb{Z}\}$ of i.i.d. standard Fréchet variables, i.e. $P(U_{ji} \leq u) = \exp(-u^{-1})$, $u > 0$. Further, $a_{jk} \geq 0$, $j \in \{1,\dots,J\}$, $k \in \mathbb{Z}$, and $\sum_{j=1}^J\sum_{k\in\mathbb{Z}} a_{jk} = 1$ such that the marginal distributions of the M_3 processes are also standard Fréchet. Note that we obtain (27) from (22) by choosing

$$g_0(x) = J\sum_{j=1}^J\sum_{k\in\mathbb{Z}} a_{jk}\mathbf{1}(x \in k + J^{-1}[j-1,j)), \quad x \in \mathbb{R}. \tag{28}$$

We will consider the following useful classification of M_3 processes. To this end, by $\mathcal{M}_\iota$ we will denote the set of all M_3 processes with $J \leq \iota \in \mathbb{N}\cup\{\infty\}$. Note that for their special structure the elements of $\mathcal{M}_1$ are canonically referred to as moving maxima or M_2 processes. Further, we put $\mathcal{M}_{\iota,n}$ for the restriction of $\mathcal{M}_\iota$ to processes up to range n. The extremal coefficient function $\phi_M(h)$ using (23) and (28) equals $\phi_M(h) = (d_M(h)+2)/2$ where

$$d_M(h) = \sum_{j=1}^{J} \sum_{k\in\mathbb{Z}} |a_{jk} - a_{j,k+h}|, \quad h \in \mathbb{Z}, M \in \mathcal{M}_\infty. \tag{29}$$

For later reference, by $\mathcal{D}_{\iota,n}$ we will denote accordingly the set of functions $d_M(h)$, $h \in \mathbb{Z}$, for all $M \in \mathcal{M}_{\iota,n}$, $\iota, n \in \mathbb{N}$.

4 A class of simple processes for given extremal coefficients

In the following we will turn the results for set correlation functions obtained in Section 2 into the construction of actual max-stable processes corresponding to given extremal coefficient functions. In particular, we will assign to each vertex of $\mathcal{F}^*_{n,\mathbb{Z}}$ (see also Definition 1.8.1 in [22]) a simple class of M_2 processes that represents the respective vertex extremal coefficient functions, cf. Corollary 1 below. We will then focus on weighted maxima of those classes in order to incorporate the convexity of $\Phi_{n,\mathbb{Z}}$. To this end, consider the following sparse class $Z(\zeta) \subseteq \mathcal{M}_{|\mathcal{C}_n|,n}$ of M_3 processes. Let $\mathcal{G} = \{\zeta = (\zeta_{[b]})_{[b]\in\mathcal{C}_n} \in [0,1]^{|\mathcal{C}_n|} : \sum_{[b]\in\mathcal{C}_n} \zeta_{[b]}|[b]| = 1\}$, and for all $\zeta \in \mathcal{G}$ define

$$\begin{aligned} Z(\zeta) = \Big\{ &(Z_t)_{t\in\mathbb{Z}} : Z_t = \max_{[b]\in\mathcal{C}_n} \zeta_{[b]} \max_{k=1}^{n} r_{[b],k} U_{[b],t-k}, t \in \mathbb{Z}, \\ &\text{and } r_{[b]} = (r_{[b],1}, \dots, r_{[b],n}) \in [b] \Big\} \end{aligned} \tag{30}$$

where as before by $\{U_{[b],i}, [b] \in \mathcal{C}_n, i \in \mathbb{Z}\}$ we denote a sequence of i.i.d. standard Fréchet variables. Note from (30) that any complete vector of representatives $r = (r_{[b]})_{[b]\in\mathcal{C}_n}$ determines a particular process $Z \in Z(\zeta)$ for any given $\zeta \in \mathcal{G}$. In the following proposition we will state an essential property of the class $Z(\zeta)$.

Proposition 4. *We have that $\phi_A(h) = \phi_B(h)$, $h \in \mathbb{Z}$, for all $A, B \in Z(\zeta)$, $\zeta \in \mathcal{G}$.*

Proof. By (30) for any fixed $Z \in Z(\zeta)$ there is a unique vector of representatives $r \in \mathcal{B}_n^{|\mathcal{C}_n|}$. We consequently find by (29) that

$$\begin{aligned} \phi_Z(h) &= \sum_{[b]\in\mathcal{C}_n} \zeta_{[b]} \sum_{k\in\mathbb{Z}} \max\left\{r_{[b],k}, r_{[b],k+h}\right\} = 2 - \sum_{[b]\in\mathcal{C}_n} \zeta_{[b]} \sum_{k\in\mathbb{Z}} \min\left\{r_{[b],k}, r_{[b],k+h}\right\} \\ &= 2 - \sum_{[b]\in\mathcal{C}_n} \zeta_{[b]} |[b]| f^*_{[b]}(h), \quad h \in \mathbb{Z}, \end{aligned}$$

where the last equality holds by (3), and where we tacitly assume that $r_{[b],k} = 0$ for all $k \in \mathbb{Z}\setminus\{1,\dots,n\}$, $[b] \in \mathcal{C}_n$. To conclude the proof note that the r.h.s. is independent of r. □

The next corollary follows immediately from the proof of Proposition 4. It identifies the abovementioned classes of M_3 processes $Z(\zeta)$ that generate the vertex extremal coefficient functions.

Corollary 1. *Let $\zeta_{[b]} = |[b]|^{-1}$ for any fixed $[b] \in \mathcal{C}_n$, and let $\zeta_{[a]} = 0$ for all $[a] \in \mathcal{C}_n$, $[a] \neq [b]$. Then, $2 - \phi_Z(h) = f^*_{[b]}(h) \in V(\mathcal{H}^*_{n,\mathbb{Z}})$ for all $Z \in Z(\zeta)$.*

Referring to Corollary 1 we shall in the following denote the vertex extremal coefficient functions by $\phi_{[b]}(h) = \phi_Z(h) = 2 - f^*_{[b]}(h)$ for any $Z \in Z(\zeta)$ with $\zeta_{[b]} = |[b]|^{-1}$, $[b] \in \mathcal{C}_n$. The functions are given in Fig. 1 for the case $n = 5$. We will show in Corollary 2 below that the restriction to the class $Z(\zeta)$, $\zeta \in \mathcal{G}$, is admissible in order to represent any extremal coefficient function $\phi \in \Phi_{n,\mathbb{Z}}$. An actual example for the reconstruction of processes based on the classes $Z(\zeta)$ will be discussed in more detail in Section 5.3.

Corollary 2. *For any extremal coefficient function $\phi \in \Phi_{n,\mathbb{Z}}$ there is a $\zeta \in \mathcal{G}$ with $|\{\zeta_{[b]} : \zeta_{[b]} > 0, [b] \in \mathcal{C}_n\}| \leq n$ such that for all $Z \in Z(\zeta)$ we have $\phi_Z(h) = \phi(h)$, $h \in \mathbb{Z}$.*

Proof. By Theorem 2 there is $S \in \sigma_n$ such that $\phi(h) = 2 - f^*_S(h)$, $h \in \mathbb{Z}$. Further, Theorem 1 with $\zeta_{[b]} = \mu_{[b]}/|[b]|$, $[b] \in \mathcal{X}$, yields that

$$f^*_S(h) = \sum_{[b]\in\mathcal{X}} f^*_{[b]}(h)\zeta_{[b]}|[b]| = 2 - \phi_Z(h) = 2 - \sum_{[b]\in\mathcal{X}} \phi_{[b]}\zeta_{[b]}|[b]|, \quad h \in \mathbb{Z},$$

for any process $Z \in Z(\zeta)$. Here, the second equality follows from the proof of Proposition 4 and the third equality is immediate from the definition of $\phi_{[b]}$. $\square$

Finally, it will be instructive to recall that any vertex extremal coefficient function $\phi_{[b]}$ reflects a class $[b] \in \mathcal{C}_n$ of homometric vectors rather than a unique vector $b \in \mathcal{B}_n$. In particular, the signature pattern [28] of a process is in general not determined by the extremal coefficient function. Even for a given function $\phi_Z \in \Phi_{n,\mathbb{Z}}$ where $Z \in Z(\zeta)$, $\zeta \in \mathcal{G}$, the signature pattern corresponding to Z is at best determined up to homometry, cf. Section 2.

5 Examples

5.1 Simplification of arbitrary M_3 processes with given coefficients

Let $A \in \mathcal{M}_{J,n}$, $J, n \in \mathbb{N}$, be given by the coefficients $a_{jk} \geq 0$, $j \in \{1, \ldots, J\}$, $k \in \mathbb{Z}$. Due to the bounded range n of A we may assume without loss of generality that $a_{jk} = 0$, $j \in \{1, \ldots, J\}$, $k \in \mathbb{Z} \setminus \{1, \ldots, n\}$. Define

$$\begin{aligned} \psi : [0,1]^n \quad &\rightarrow \quad [0,1]^n \\ b \quad &\mapsto \quad \max\left(b - \min\{b_i : b_i > 0\}, 0\right). \end{aligned}$$

Let the M_3 process $C \in \mathcal{M}_{(Jn),n}$ carry a third index l in addition to j, k, and let C be defined by the coefficients $c_{jlk} \geq 0$, $j \in \{1, \ldots, J\}$, $l, k \in \{1, \ldots, n\}$, that is

$$C_t = \max_{j=1}^{J} \max_{l=1}^{n} \max_{k\in\mathbb{Z}} c_{jlk} U_{jl,t-k}, \quad t \in \mathbb{Z}, \tag{31}$$

where the sequence $\{U_{jli}, j \in \{1, \ldots, J\}, l \in \{1, \ldots, n\}, i \in \mathbb{Z}\}$ again represents i.i.d. standard Fréchet variables. Further, let $c_{jl} = (c_{jl,1}, \ldots, c_{jln}) = \psi^{l-1}(a_j) - \psi^l(a_j)$ where $a_j = (a_{j,1}, \ldots, a_{jn})$ are the coefficients of A, and $\psi^l = \psi \circ \ldots \circ \psi$ gives the l-fold composition of ψ. We will make use of the following simple fact.

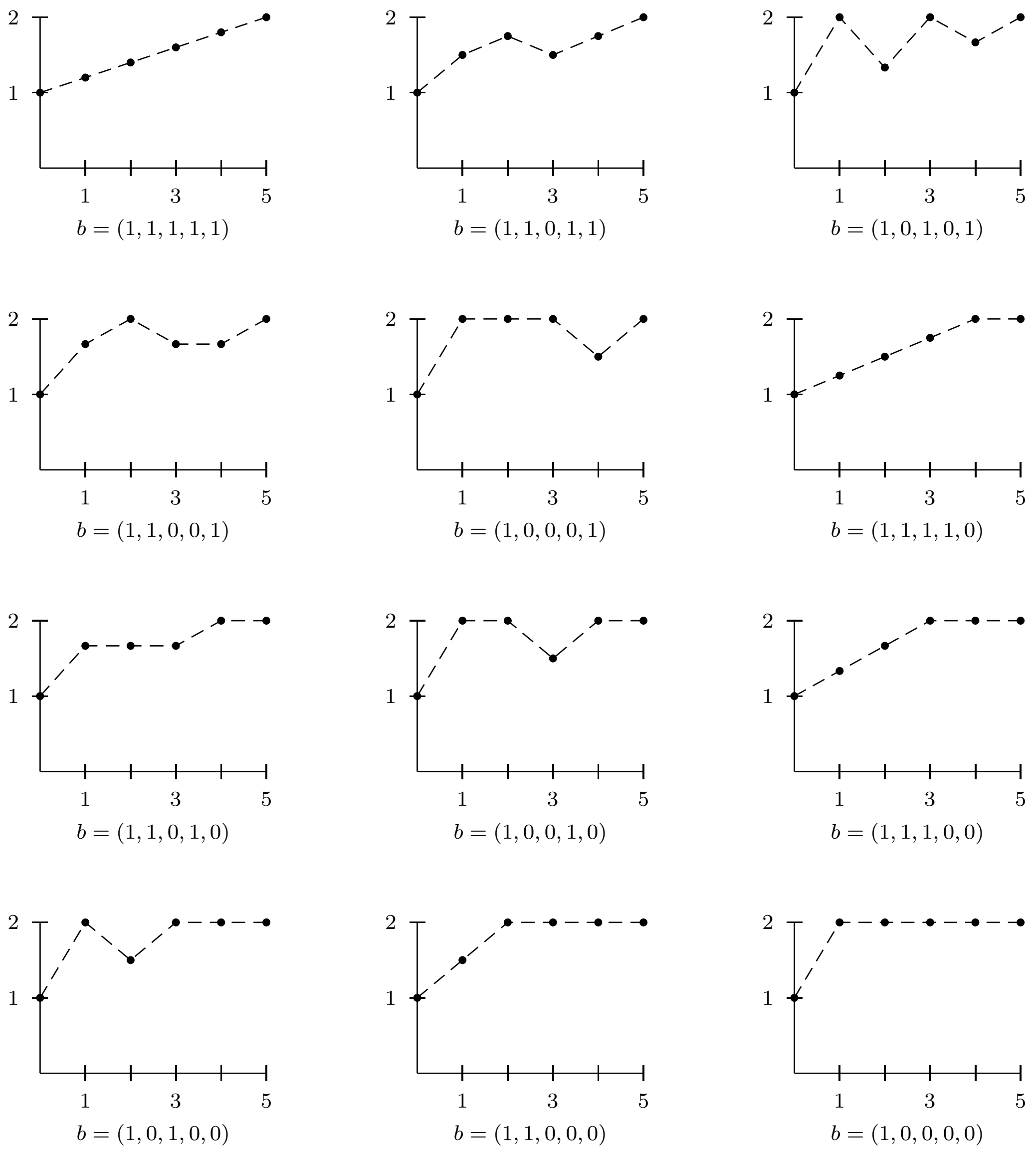

Figure 1: Vertex extremal coefficient functions $\phi_{[b]}(h)$, $[b] \in \mathcal{C}_n$, for $n = 5$ and $h = 0, \ldots, 5$. The respective equivalence classes are identified by the corresponding representatives $b \in \mathcal{B}_n$. Note that we only tentatively include the lines joining the points as we confine our analysis to $\mathbb{Z}$.

Lemma 3. *For all $a_1, a_2, m \in \mathbb{R}$ let $b_1 = \max(a_1 - m, 0)$, $b_2 = \max(a_2 - m, 0)$, $c_1 = \min(a_1, m)$ and $c_2 = \min(a_2, m)$. Then $|a_1 - a_2| = |b_1 - b_2| + |c_1 - c_2|$.*

Now, by a repeated application of Lemma 3 it follows that

$$d_A(h) = d_C(h) = \sum_{j=1}^{J} \sum_{l=1}^{n} \sum_{k=1}^{n} |c_{jlk} - c_{jl,k+h}|, \quad h \in \mathbb{Z}.$$

We will finally emphasize that the vertex extremal coefficient functions may be identified naturally from C. To this end, for $c_{jl} \neq 0$ let $m_{jl} = \max_k c_{jlk}$ and $\hat{c}_{jl} = c_{jl}/m_{jl}$ such that by definition of c_{jl} we have $\hat{c}_{jl} \in \mathcal{B}_n$ for all $j \in \{1, \ldots, J\}$ and all $l \in \{1, \ldots, n\}$. Next, put

$$C_{jl,t} = |[\hat{c}_{jl}]|^{-1} \max_{k \in \mathbb{Z}} \hat{c}_{jlk} U_{jl,t-k} \in \mathcal{M}_{1,n}, \quad t \in \mathbb{Z},$$

such that (29) yields $\phi_{C_{jl}} = |[\hat{c}_{jl}]|^{-1} \sum_{k \in \mathbb{Z}} \max\{\hat{c}_{jlk}, \hat{c}_{jl,k+h}\}$. Further, using (31) we get that $C_t = \max_{j=1}^{J} \max_{l=1}^{n} |[\hat{c}_{jl}]| m_{jl} C_{jl,t}$, $t \in \mathbb{Z}$, and, accordingly, by (29) we now have

$$\phi_C(h) = \sum_{j=1}^{J} \sum_{l=1}^{n} m_{jl} |[\hat{c}_{jl}]| \phi_{C_{jl}}(h). \tag{32}$$

The fact that $\hat{c}_{jl} \in \mathcal{B}_n$ yields by (30) that $C_{jl} \in Z(\zeta)$ for $\zeta_{[\hat{c}_{jl}]} = |[\hat{c}_{jl}]|^{-1}$ such that for all processes C_{jl} with $\hat{c}_{jl} \in [b]$, $[b] \in \mathcal{C}_n$, we find by Corollary 1 that $\phi_{C_{jl}} = \phi_{[b]}$. Finally, (32) gives

$$\phi_A(h) = \phi_C(h) = \sum_{[b] \in \mathcal{C}_n} \beta_{[b]} \phi_{[b]}, \quad h \in \mathbb{Z},$$

where $\beta_{[b]} = \sum_{j=1}^{J} \sum_{l=1}^{n} m_{jl} |[\hat{c}_{jl}]| \mathbf{1}(\hat{c}_{jl} \in [b])$ for all $[b] \in \mathcal{C}_n$. To conclude the example note that applying the arguments discussed in Sections 2 and 4 we may further reduce the appropriate index set to $\mathcal{X} \subseteq \mathcal{C}_n$.

5.2 Blind reconstruction of M_2 processes

We shall now turn to the blind retrieval of an actual example process for an extremal coefficient function of a max-stable process in discrete time with finite range n. Here, we will first restrict to the class of M_2 processes, that is we put $J = 1$, in order to show that given a priori knowledge about J there are alternative approaches for the reconstruction of processes that do not necessarily resort to Corollary 2. Below we shall discuss such an approach. To this end, let $d_X \in \mathcal{D}_{1,n}$ be given. Then, there is an unknown (not necessarily unique) M_2 process X that is determined by its coefficients $x_1, \ldots, x_n$ such that by (29) we have

$$d_X(h) = \sum_{k=1}^{2n} |x_k - x_{k+h}|, \quad h = 1, \ldots, n. \tag{33}$$

In order to turn (33) into tractable systems of linear equations we will make use of the following lemma that can be easily seen.

Lemma 4. *Let $x_i \geq 0$, $i = 1, \ldots, n$, and $x_i = 0$, else. There is a permutation π on $\{1, \ldots, n\}$ such that $x_{\pi^{-1}(1)} \geq \ldots \geq x_{\pi^{-1}(n)}$ and*

$$\sum_{i \in \mathbb{Z}} |x_i - x_{i-h}| = \sum_{i=1}^{n} \alpha_{\pi,h,i} x_i, \quad h = 1, \ldots, n, \tag{34}$$

where

$$\alpha_{\pi,h,i} = 2\left[\mathbf{1}(\pi(i) < \pi(i-h)) + \mathbf{1}(\pi(i) < \pi(i+h)) - 1\right] \in \{-2, 0, 2\} \tag{35}$$

for all $h, i \in \{1, \ldots, n\}$, and $\pi(i) = \infty$ for all $i \in \mathbb{Z} \setminus \{1, \ldots, n\}$. Further, $\sum_{i=1}^{n} \alpha_{\pi,h,i} = 2h$, $h = 1, \ldots, n$. The sequence of coefficients $\alpha_{\pi,h,i}$, $h, i = 1, \ldots, n$, is unique for a given permutation π, and vice versa.

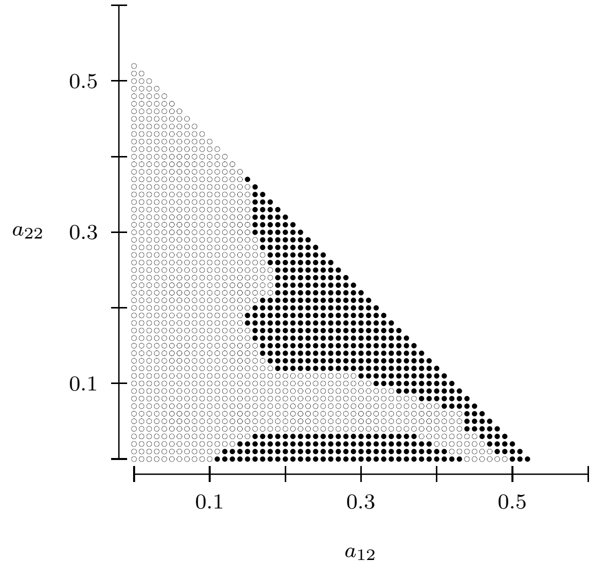

Figure 2: Admissible combinations of a_{12} and a_{22} for the process A discussed in Section 5.3 where $d_A(h) \in \mathcal{D}_{1,n}$ ($\circ$) and $d_A(h) \notin \mathcal{D}_{1,n}$ ($\bullet$).

For the unknown M_2 process X according to Lemma 4 there is a (not necessarily unique) permutation π such that $x_{\pi^{-1}(1)} \geq \ldots \geq x_{\pi^{-1}(n)}$ and such that by (33) and (34) we have

$$d_X(h) = \sum_{i=1}^{n} \alpha_{\pi,h,i} x_i, \quad h = 1, \ldots, n. \tag{36}$$

As π is unknown so is the sequence $\alpha_{\pi,h,i}$, $h, i = 1, \ldots, n$, and hence running through all possible permutations Eq. (36) represents $n!$ systems of linear equations where in each case the coefficients $\alpha_{\pi,h,i}$ are given by (35). Consequently, an appropriate permutation π is associated to at least one of the linear systems, and a corresponding solution $x_1, \ldots, x_n$ representing a process X exists. The latter can be found for instance via a linear program [2]. Note also that for any $d \in \mathcal{D}_{\infty,n}$ the above approach reveals whether any solution to (36) exists at all, i.e. whether $d \in \mathcal{D}_{1,n} \subseteq D_{\infty,n}$.

5.3 Blind reconstruction of M_3 processes

As indicated by the above discussion we find that even with respect to the function $d_A(h)$ for an arbitrary process $A \in \mathcal{M}_{2,n}$ it appears to be nontrivial to state whether $d_A(h) \in \mathcal{D}_{1,n}$. Except for some pathological examples we are not aware of a suitable analytic criterion that focusses directly on the coefficients of A. Thus, using (36) and the method outlined above we will check by a trial and error procedure whether for simulated processes $A \in \mathcal{M}_{2,n}$ with arbitrary coefficients a_{jk}, $j \in \{1, 2\}$, $k \in \{1, \ldots, 5\}$, we have that $d_A(h) \in \mathcal{D}_{1,n}$. We give particular such processes $A(a_{12}, a_{22})$ where $d_{A(a_{12},a_{22})}(h) \notin \mathcal{D}_{1,n}$ for at least some a_{12}, a_{22} in Table 2. In order to get more insight into the sensitivity of our results to changes of the coefficients we run through all admissible values of a_{12} and a_{22} with all other coefficients fixed and state whether $d_{A(a_{12},a_{22})} \in \mathcal{D}_{1,n}$. The result is given in Fig. 2. Apart from a certain

a_{jk}	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$j=1$	0.01	a_{12}	0.02	0.05	0.21
$j=2$	$0.52-a_{12}-a_{22}$	a_{22}	0.12	0.06	0.01

Table 2: Coefficients of the process $A(a_{12}, a_{22})$ discussed in Section 5.3.

tradeoff between a_{12} and a_{22} along the upper right boundary there appears to be no definite structure evident from the figure.

Next, we will discuss an example for the reconstruction of max-stable processes that makes use of Corollary 2, that is we do not consider the above instances where $J=1$. We put $n=5$ in order to cover at the same time the case $|\mathcal{B}_n/{\sim_h}| > |\mathcal{C}_n|$ discussed in Section 2. To this end, from the class of processes A discussed above we arbitrarily choose $A(0.15, 0.18)$ with $d_{A(0.15,0.18)}(h) \notin \mathcal{D}_{1,n}$ where, in particular,

h	1	2	3	4	5
$d_{A(0.15,0.18)}(h)$	1.06	1.46	1.54	1.96	2.00

From now on, we will assume $d(h) = d_A(h)$ to be given and consider the process $A \in \mathcal{M}_{2,n} \setminus \mathcal{M}_{1,n}$ to be unknown. Let $\mathcal{G}_d = \{\zeta \in \mathcal{G} : d_Z(h) = d(h), h \in \mathbb{Z}, Z \in Z(\zeta)\}$ be the set of all vectors $\zeta \in \mathcal{G}$ that determine sets $Z(\zeta)$ of suitable candidate processes. Note that $\mathcal{G}_d$ is nonempty by Corollary 2. We will focus on the following system of linear equations

$$d(h) = d_Z(h), \quad Z \in Z(\zeta), h \in \mathbb{Z}, \tag{37}$$

where by Proposition 4 we may choose $Z \in Z(\zeta)$ arbitrarily. A particular process $Z \in Z(\zeta)$ is given in Table 3. To simplify notation we shall replace the indices $[b]$, $[b] \in \mathcal{C}_n$, by $1, \ldots, 12$ according to the second column in Tab. 3. We now get from (37) and (29) for Z as in Tab. 3

b	[b]	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$(1,1,1,1,1)$	1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1
$(1,1,0,1,1)$	2	ζ_2	ζ_2		ζ_2	ζ_2
$(1,0,1,0,1)$	3	ζ_3		ζ_3		ζ_3
$(1,1,0,0,1)$	4	ζ_4	ζ_4			ζ_4
$(1,0,0,0,1)$	5	ζ_5				ζ_5
$(1,1,1,1,0)$	6	ζ_6	ζ_6	ζ_6	ζ_6	
$(1,1,0,1,0)$	7	ζ_7	ζ_7		ζ_7	
$(1,0,0,1,0)$	8	ζ_8			ζ_8	
$(1,1,1,0,0)$	9	ζ_9	ζ_9	ζ_9		
$(1,0,1,0,0)$	10	ζ_{10}		ζ_{10}		
$(1,1,0,0,0)$	11	ζ_{11}	ζ_{11}			
$(1,0,0,0,0)$	12	ζ_{12}				

Table 3: Example coefficients $\zeta_{[b]} r_{[b],k}$, $k = 1, \ldots, 5$, for a specific process $Z \in Z(\zeta) \subseteq \mathcal{M}_{12,5}$, cf. (30). Here, $(\zeta_1, \ldots, \zeta_{12}) = \zeta$ where we use the notational convention explained after (37). See Fig. 1 for an illustration of the vertex extremal coefficient functions $\phi_{[b]}$, $[b] \in \mathcal{C}_n$, that are retrievable from any $Z \in Z(\zeta)$ if $\zeta_{[b]} = |[b]|^{-1}$. Note also that the case $b = (1,1,1,0,1)$ is not included in the table as $[b] \in \mathcal{B}_n/{\sim_h}$ but $[b] \notin \mathcal{C}_n$.

that

$$\begin{aligned}
d_Z(1) =&\ 2\zeta_1 + 4\zeta_2 + 6\zeta_3 + 4\zeta_4 + 4\zeta_5 + 2\zeta_6 + 4\zeta_7 + 4\zeta_8 + 2\zeta_9 + 4\zeta_{10} + 2\zeta_{11} + 2\zeta_{12}\\
d_Z(2) =&\ 4\zeta_1 + 6\zeta_2 + 2\zeta_3 + 6\zeta_4 + 4\zeta_5 + 4\zeta_6 + 4\zeta_7 + 4\zeta_8 + 4\zeta_9 + 2\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}\\
d_Z(3) =&\ 6\zeta_1 + 4\zeta_2 + 6\zeta_3 + 4\zeta_4 + 4\zeta_5 + 6\zeta_6 + 4\zeta_7 + 2\zeta_8 + 6\zeta_9 + 4\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}\\
d_Z(4) =&\ 8\zeta_1 + 6\zeta_2 + 4\zeta_3 + 4\zeta_4 + 2\zeta_5 + 8\zeta_6 + 6\zeta_7 + 4\zeta_8 + 6\zeta_9 + 4\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}\\
d_Z(5) =&10\zeta_1 + 8\zeta_2 + 6\zeta_3 + 6\zeta_4 + 4\zeta_5 + 8\zeta_6 + 6\zeta_7 + 4\zeta_8 + 6\zeta_9 + 4\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}.
\end{aligned}$$

Numerically, if $\phi(h)$ is a valid extremal coefficient function, i.e. $\mathcal{G}_d$ is nonempty, a particular element $\zeta \in \mathcal{G}_d$ may be determined by expanding (37) to a linear program. Here, using [2] we find e.g.

$$\zeta = (\zeta_1, \dots, \zeta_{12}) = (0.020, 0, 0, 0, 0, 0.085, 0, 0.105, 0, 0.040, 0.135, 0)$$

as a valid (not necessarily unique) solution. We point out that according to Corollary 2 there are $n = 5$ nonzero elements in ζ.

Remark 1. For all processes $Z \in Z(\zeta)$, $\zeta \in \mathcal{G}$, we have that $\sum_{[b]\in\mathcal{C}_n} \zeta_{[b]} = \theta_Z$ where θ_Z denotes the extremal index, a measure for the expected cluster size of Z, cf. [13]. Note also that $1/n \leq \theta_Z \leq 1$, i.e. the range n of Z imposes a lower bound on the extremal index.

5.4 Necessary conditions for valid extremal coefficient functions

Apart from the reconstruction of max-stable processes for given extremal coefficient functions the technique applied in Section 5.3 is applicable also to evaluate whether a supposed extremal coefficient function of order n is valid for max-stable processes on $\mathbb{Z}$. To our knowledge, in the literature so far only necessary conditions for extremal coefficient functions to be admissible have been discussed [5, 24]. Linking the results for first order variograms (madograms) discussed by [16] to extremal coefficient functions it is shown in [5] that every valid extremal coefficient function $\phi(h)$ for all $h, k \in \mathbb{R}$ satisfies

$$\begin{aligned}
\phi(h+k) &\leq \phi(h)\phi(k), && \text{(C1)}\\
\phi(h+k)^\tau &\leq \phi(h)^\tau + \phi(k)^\tau - 1, \quad 0 \leq \tau \leq 1, && \text{(C2)}\\
\phi(h+k)^\tau &\geq \phi(h)^\tau + \phi(k)^\tau - 1, \quad \tau \leq 0. && \text{(C3)}
\end{aligned}$$

In addition, it is well-known that $\phi(h)$ is positive semi-definite, cf. [24]. We give an example showing that conditions (C1) to (C3) are indeed not sufficient. The construction of such an example is not evident but substantially facilitated by knowledge of the vertex extremal coefficient functions displayed in Fig. 1. Consider e.g. the following function $p : \mathbb{Z} \to [1, 2]$, $p(-h) = p(h)$, with

h	0	1	2	3	4
$p(h)$	1	5/3	5/3	3/2	2

and $p(h) = 2$, $h \geq 5$. Note that

$$p(x) = \phi_{[b]}(x) \quad \text{for } x \in \{0, 3, 4, 5\} \tag{38}$$

and that

$$p(x) \neq \phi_{[b]}(x) \quad \text{for } x \in \{1, 2\} \tag{39}$$

for $b = (1,0,0,1,0) \in \mathcal{B}_n$. Further, by Fig. 1 we easily find that

$$\big(\phi_{[b]}(3), \phi_{[b]}(4), \phi_{[b]}(5)\big) \neq \sum_{[a] \in \mathcal{C}_n \setminus \{[b]\}} \big(\phi_{[a]}(3), \phi_{[a]}(4), \phi_{[a]}(5)\big)\, \mu_{[a]} \tag{40}$$

for any $\mu_{[a]} \in [0,1]$, $[a] \in \mathcal{C}_n \setminus \{[b]\}$. Now, using (38) to (40) we get from the convexity of $\Phi_{n,\mathbb{Z}}$ that p is not a valid extremal coefficient function. However, it is readily verified that p still satisfies (C1) to (C3).

6 Restrictions on the range of extremal coefficient functions

In the following we will study a lower bound on the range of a max-stable process if the corresponding extremal coefficient is known for a fixed $h \in \mathbb{N}$ only. More precisely, if for any fixed $h \in \mathbb{N}$ the extremal coefficient $\phi(h)$ is given we will specify the smallest lag $\bar{h} \geq h$ for which $\phi(\tilde{h}) = 2$ for all $\tilde{h} \geq \bar{h}$ is possible. In practice, the approach will be applicable to the study of the actual (bounded) memory spread of short memory processes, cf. Definition 1. Consider for instance the question of a lower bound on the memory of financial markets after shocks when information is limited to estimates of a single extremal coefficient.

Theorem 3. *Let $\phi_Y(h) \in [1,2)$ be given for some fixed $h \in \mathbb{N}$ and some max-stable process $Y \in \mathcal{M}_\infty$. We have that $Y \notin \mathcal{M}_{\infty, r_\phi}$ for any*

$$r_\phi \in \begin{cases} \mathbb{N} & \text{if } \phi(h) = 1, \\ \{1, \ldots, [\![(\phi(h)-1)^{-1}]\!]h\}, & \text{else,} \end{cases}$$

where $[\![x]\!] = \max\{n \in \mathbb{Z} : n < x\}$ for any $x \in \mathbb{R}$. On the other hand, if $\phi(h) \in (1,2)$, for some $h \in \mathbb{N}$, then a process $Y \in \mathcal{M}_{\infty, [\![(\phi(h)-1)^{-1}]\!]h+1}$ with $\phi_Y(h) = \phi(h)$ exists.

Proof. The assertion for $\phi(h) = 1$, $h > 0$, follows directly from Theorem 1.4.1(2) in [22]. The proof for $\phi(h) \in (1,2)$ is based on the M_3 representation for max-stable processes discussed in Section 3, and comprises three steps. First, within the classes $\mathcal{M}_{\infty,K+h-1}$ of all M_3 processes with maximum range $K+h-1$, $K \in \mathbb{N}h+1 = \{h+1, 2h+1, \ldots\}$, we will define a simple M_3 process $A_{K,h} \in \mathcal{M}_{1,K}$ of range K. Then, we will show that $A_{K,h} \in \mathcal{M}_{\infty,K+h-1}$ minimizes $\phi_B(h)$ for all $B \in \mathcal{M}_{\infty,K+h-1}$. Based on this finding we may conclude in step three that all processes $Z \in \mathcal{M}_\infty$ with $\phi_Z(h) = \phi(h)$ are at least of range $[\![(\phi(h)-1)^{-1}]\!]h+1$. We give an example in order to show that the bounds are sharp.

1. For any $K \in \mathbb{N}h+1$ let the process $A_{K,h} \in \mathcal{M}_{1,K}$ be given by the coefficients $a_{K,k}$, $k \in \mathbb{Z}$, where

$$a_{K,ih+1} = \left(\frac{K-1}{h} + 1\right)^{-1}, \quad i \in \{0, 1, \ldots, (K-1)/h\}, \tag{41}$$

and all other coefficients zero. In particular, by (29) we have

$$d_{A_{K,h}}(h) = 2a_{K,1}. \tag{42}$$

Without loss of generality we let $B \in \mathcal{M}_{\infty,K+h-1}$ be given by

$$0 \leq b_{1k} = a_{K,k} + \varepsilon_{1k} \leq 1, \quad k \in \{1, \ldots, K+h-1\}, \tag{43}$$

where the $a_{K,k}$ are chosen according to (41). Further, for $j \in \{2, 3, \ldots\}$ and $k \in \{1, \ldots, K + h - 1\}$ we let

$$0 \leq b_{jk} = \varepsilon_{jk} \leq 1 \tag{44}$$

and tacitly assume all other coefficients to be zero. Now, from $\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} b_{jk} = 1$ we get by (43) and (44) that

$$\sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \varepsilon_{jk} = 0 \tag{45}$$

and, in particular,

$$\sum_{i=0}^{\frac{K-1}{h}} \varepsilon_{1,ih+1} \leq 0. \tag{46}$$

2. We show that for all processes $B \in \mathcal{M}_{\infty, K+h-1}$ it holds that

$$d_B(h) \geq d_{A_{K,h}}(h), \quad K \in \mathbb{N}h + 1. \tag{47}$$

To this end, note that by (29) we find that (47) is equivalent to

$$-\varepsilon_{11} - \varepsilon_{1,K} \leq \sum_{j=1}^{\infty} \sum_{l=1}^{h} \sum_{i=0}^{\frac{K-1}{h}-1} |\varepsilon_{j,l+(i+1)h} - \varepsilon_{j,l+ih}| + \sum_{j=1}^{\infty} \sum_{\substack{l=1 \\ j+l>2}}^{h} (\varepsilon_{jl} + \varepsilon_{j,K+l-1}). \tag{48}$$

Now, (43) and (44) yield that $\varepsilon_{j,l}, \varepsilon_{j,K+l-1} \geq 0$ for all $j \in \mathbb{N}$ and $l \in \{1, \ldots, h\}$ with $j + l > 2$, such that (48) holds if $\min\{\varepsilon_{11}, \varepsilon_{1,K}\} \geq 0$. In order to show (48) for the case $\min\{\varepsilon_{11}, \varepsilon_{1,K}\} < 0$ put $N = \{ih + 1,\ i = 0, 1, \ldots, (K-1)/h\}$ and for $j \in \mathbb{N}$, $l \in \{1, 2, \ldots, h\}$ let $S_{jl} = \sum_{i \in N+l-1} \varepsilon_{ji}$, $\bar{\mu}_{jl} = S_{jl}|N|^{-1}$ and $\mu_{jl,\max} = \max_{i \in N+l-1} \varepsilon_{ji}$. Further, let

$$\mu_{1,\min} = -\min\{\mu_{1,1,\max}, 0\}. \tag{49}$$

Now, we find that

$$\begin{aligned} -\varepsilon_{11} - \varepsilon_{1,K} &\leq |\varepsilon_{11}| + |\varepsilon_{1,K}| \leq |\varepsilon_{11} - \mu_{1,\min}| + |\varepsilon_{1,K} - \mu_{1,\min}| + 2\mu_{1,\min} \\ &\leq \sum_{i=0}^{\frac{K-1}{h}-1} |\varepsilon_{1,ih+1} - \varepsilon_{1,(i+1)h+1}| + 2\mu_{1,\min}. \end{aligned} \tag{50}$$

Also, by (49) we get for $\min\{\varepsilon_{11}, \varepsilon_{1,K}\} < 0$ that $\min_{i \in N} \varepsilon_{1,i} \leq \mu_{1,\min} \leq \max_{i \in N} \varepsilon_{1,i}$ which yields the second inequality in (50). Next, we have

$$\mu_{1,\min} \leq \frac{-S_{1,1}}{|N|} = \frac{1}{|N|} \sum_{j=1}^{\infty} \sum_{\substack{l=1 \\ j+l>2}}^{h} S_{j,l} = \sum_{j=1}^{\infty} \sum_{\substack{l=1 \\ l+j>2}}^{h} \bar{\mu}_{jl} \leq \sum_{j=1}^{\infty} \sum_{\substack{l=1 \\ j+l>2}}^{h} \mu_{j,l,\max}. \tag{51}$$

Here, if $\max_{i \in N}\{\varepsilon_{1,i}\} \geq 0$ note that $\mu_{1,\min} = 0$, such that the first inequality is obvious from the fact that $S_{1,1} \leq 0$ by (46). Else, if $\max_{i \in N}\{\varepsilon_{1,i}\} < 0$ we get that $\mu_{1,\min} = \min_{i \in N}\{-\varepsilon_{1,i}\} \leq |N|^{-1} \sum_{i \in N} -\varepsilon_{1,i} = -S_{1,1}|N|^{-1}$ which in that case yields the first inequality. Further, the first equality in (51) holds by (45) and the second equality as well as the second inequality are immediate. Finally, for all $j \in \mathbb{N}$ and all $l \in \{1, \dots, h\}$ with $l + j > 2$ we have

$$\begin{aligned} 2\mu_{j,l,\max} &= |\mu_{j,l,\max} - \varepsilon_{jl}| + |\mu_{j,l,\max} - \varepsilon_{j,l+K-1}| + \varepsilon_{jl} + \varepsilon_{j,l+K-1} \\ &\leq \sum_{i=0}^{\frac{K-1}{h}-1} |\varepsilon_{j,l+(i+1)h} - \varepsilon_{j,l+ih}| + \varepsilon_{jl} + \varepsilon_{j,l+K-1}. \end{aligned} \tag{52}$$

where the equality follows from the fact that $0 \leq \varepsilon_{jl} \leq \mu_{jl,\max}$ for $l + j > 2$. Now, (48) holds by (50) to (52).

3. Let $\mathcal{Z}(d(h)) \subseteq \mathcal{M}_\infty$ be the class of all M_3 processes Z with $d_Z(h) = d(h)$. By (47) it follows that $\mathcal{Z}(d(h)) \cap \mathcal{M}_{\infty,\kappa+h-1} = \emptyset$ for all

$$\kappa \in \{K \in \mathbb{N}h + 1 : d_{A_{K,h}}(h) > d(h)\} = \{K \in \mathbb{N}h + 1 : K < [\![2/d(h)]\!]h + 1\}$$

where (42) yields the equality. Let $K^* = [\![2/d(h)]\!]h + 1$ where $K^* < \infty$ from the fact that $d(h) > 0$ by assumption. In particular, we now have that

$$d_{A_{K^*,h}}(h) \leq d(h) < d_{A_{K^*-1,h}}(h). \tag{53}$$

It remains to show that a process $Z^* \in \mathcal{M}_{\infty,K^*} \cap \mathcal{Z}(d(h))$ exists. To this end, let Z^* be given by $z_k^* = a_{K^*,k} - \varepsilon_k + \delta_k$, $k \in \{1, 2, \dots, K^*\}$ where $a_{K^*,k}$ are the coefficients of $A_{K^*,h}$, cf. (41). Further, we put $\varepsilon_{ih+1} = \frac{1}{2}a_{K^*,1}(d(h) - 2a_{K^*,1})$, $i \in \{0, 1, \dots, (K^*-1)/h\}$, $\delta_2 = \frac{1}{2}(d(h) - 2z_1^*)$ and all other coefficients zero such that Z^* is of range K^*. Note that (53) yields

$$0 \leq \frac{1}{2}d(h) - a_{K^*,1} < a_{K^*-1,1} - a_{K^*,1} = \frac{h}{K^* - 1} - \frac{h}{K^* - 1 + h} < 1$$

such that $0 \leq \varepsilon_{1+ih} < a_{K^*,1}$, $i \in \{0, 1, \dots, (K^* - 1)/h\}$. Further, using (53) we have

$$2z_1^* = 2(a_{K^*,1} - \varepsilon_1) < 2a_{K^*,1} = d_{A_{K^*,h}}(h) \leq d(h)$$

which yields that $\delta_2 > 0$. Finally, $d_{Z^*}(h) = d(h)$ is a consequence of (29).

□

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References

[1] J. Beran. Statistics for long-memory processes. Chapman & Hall, New York, 1994.

[2] Michel Berkelaar et al. lpSolve: Interface to Lp_solve v. 5.5 to solve linear/integer programs, 2008. R package version 5.6.4.

[3] C. Caratheodory. Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. Rend. Circ. Mat. Palermo, 32:193–217, 1911.

[4] J.-P. Chilès and P. Delfiner. Geostatistics. Modeling Spatial Uncertainty. John Wiley & Sons, New York, Chichester, 1999.

[5] D. Cooley, P. Naveau, and P. Poncet. Variograms for spatial max-stable random fields. In P. Bertail, P. Doukhan, and P. Soulier, editors, Dependence in Probability and Statistics, pages 373–390. Springer, New York, 2006.

[6] R.A. Davis and T. Mikosch. The extremogram: A correlogram for extreme events. Bernoulli, 15(4):977–1009, 2009.

[7] R.A. Davis and S.I. Resnick. Prediction of stationary max-stable processes. Ann. Appl. Probab., 3:497–525, 1993.

[8] L. de Haan and J. Pickands. Stationary min-stable stochastic processes. Probab. Th. Rel. Fields., 72:477–492, 1986.

[9] A. Ehlert and M. Schlather. A Constructive Proof for the Extremal Coefficient of a Dissipative Max-Stable Process on $\mathbb{Z}$ being a Set Covariance. Discussion Paper No. 31, Courant Research Centre – PEG, University of Goettingen, 2010.

[10] V. Fasen, C. Klüppelberg, and M. Schlather. High-level dependence in time series models. Extremes, 13(1):1–33, 2008.

[11] T. Gneiting. Nonseparable, stationary covariance functions for space-time data. J. Amer. Statist. Assoc., 97:590–600, 2002.

[12] L.J. Guibas and M. Odlyzko. Periods in strings. Journal of Combinatorial Theory, Series A, 30(1):19–42, 1981.

[13] M.R. Leadbetter. Extremes and local dependence of stationary sequences. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 65:291–306, 1983.

[14] P. Lemke, S.S. Skiena, and W.D. Smith. Reconstructing sets from interpoint distances. In Discrete and Computational Geometry: The Goodman Pollack Festschrift, pages 597–631. Springer, Berlin, 2003.

[15] G. Matheron. Random Sets and Integral Geometry. John Wiley & Sons, New York, 1975.

[16] G. Matheron. Suffit-il, pour une covariance, d'être de type positif? Sci. de la Terre, Sér. Informatique Géologique, 26:51–66, 1987.

[17] I. Molchanov. Convex geometry of max-stable distributions. Extremes, 11(3):235–259, 2008.

[18] A.L. Patterson. A direct method for the determination of the components of interatomic distances in crystals. Zeitschr. Krist., 90:517–542, 1935.

[19] A.L. Patterson. Ambiguities in the x-ray analysis of crystal structures. Physical Review, 65(5-6):195–201, 1943.

[20] E. Rivals and S. Rahmann. Combinatorics of periods in strings. Journal of Combinatorial Theory, Series A, 104(1):95–113, 2003.

[21] G. Samorodnitsky. Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes. Ann. Prob., 32(2):1438–1468, 2004.

[22] Z. Sasvári. Positive Definite and Definitizable Functions. Akademie Verlag, Berlin, 1994.

[23] M. Schlather. Models for stationary max-stable random fields. Extremes, 5:33–44, 2002.

[24] M. Schlather and J.A. Tawn. A dependence measure for multivariate and spatial extreme values: Properties and inference. Biometrika, 90(1):139–156, 2003.

[25] R.L. Smith. Max-stable processes and spatial extremes. Unpublished manuscript, 1990.

[26] R.L. Smith and I. Weissman. Characterization and estimation of the multivariate extremal index. Technical report, University of North Carolina at Chapel Hill, NC, USA, 1996.

[27] Y. Wang and S.A. Stoev. On the structure and representations of max-stable processes. Tech. report, Department of Statistics, University of Michigan, 2009. arXiv:0903.3594v2.

[28] Z. Zhang and R.L. Smith. The behavior of multivariate maxima of moving maxima processes. J. Appl. Prob., 41(4):1113–1123, 2004.