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Poverty Severity in a Multidimensional Framework: The Issue of Inequality between Dimensions

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Poverty Severity in a Multidimensional Framework: The Issue of Inequality between Dimensions

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Abstract

This paper contributes to the axiomatic foundation of multidimensional poverty measures. A well-known problem in the multidimensional framework is that the identification method used in the one-dimensional framework, the union method, leads to exaggerated poverty rates. So far, this problem has been addressed by either changing the identification method itself or by introducing different weighting schemes – which all have in common that they assume attributes to be substitutes. In our paper we claim that the exaggeration problem is first of all an issue of how distribution sensitivity is accounted for and thus ought to be addressed at the aggregation instead of the identification level. In fact, we provide evidence that the way in which the Transfer principle, which accounts for distribution sensitivity in the one-dimensional framework, has been extended to the multidimensional framework is incomplete. We demonstrate that by solving this aggregation problem with the introduction of an additional axiom, the exaggeration problem at the identification level is, as a direct consequence, automatically solved as well. Finally, we derive a family of poverty measures whose specific, axiomatically implied weighting structure solves the exaggeration problem for ordinal as well as cardinal data while at the same time allowing for an independent relationship between attributes. We demonstrate that some of the most well-known poverty measures like the Multidimensional Poverty Index are special cases of this family.

JEL Classification: I32

Keywords: Multidimensional poverty measurement, axiomatic approach, aggregation of poor, poverty severity

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1 Introduction

The very first issue each poverty analysis has to address is the way poverty should be measured. Over time, experts and academics derived a variety of suggestions, the discussion, however, continues to be controversial. The main reason for all controversy is founded in the obvious trade-off between the demand for inclusiveness on one hand and that of applicability on the other hand. Poverty measures should comprise preferably every aspect of deprivation and at the same time be lean enough to be easily applicable.

One of the main controversies revolves around the issue whether poverty measurement should be based on a one- or multidimensional approach. For a long time, insufficient income was quite unanimously considered to be an appropriate and easily applicable indicator for the multidimensional character of poverty. However, this one-dimensional approach has increasingly come under criticism.

Higher income surely improves an individual's ability to fulfil his or her needs. But the underlying a priori restriction, which effectively assigns a weight of one to the income dimension and zero weights to all other potential poverty dimensions, seems to be far too constraining and implies a complete loss of information on dimension-specific shortfalls. The existence of perfect and complete markets is another strong a priori assumption of the one-dimensional approach. It presumes that a market exists for every single poverty dimension and that prices reflect the utility weights each household assigns to these dimensions. However, especially in the context of developing countries, markets are rather often imperfect or do not even exist at all. In addition to the technical objections, empirical studies cast further doubts on a close correlation between income and other dimensions of poverty. Lipton and Ravallion (1995), for instance, provide evidence that income levels are not per se important for poverty measurement but rather how the income is spent. The brief summary of the main objections against the income approach casts major doubts on its justifiability. Empirical evidence suggests that these doubts are reasonable: one- and multidimensional approaches diverge substantially with regard to the identification of the poor (e.g. Klasen 2000, Ramos 2005).

Though we of course acknowledge the obvious advantages of the income approach we nevertheless believe that sufficient evidence has been provided by now to conclude that it is inadequate. This paper therefore seeks to contribute to current research efforts to operationalise a multidimensional approach to poverty measurement. In particular, we address an anomaly in the measurement of poverty severity.

The paper proceeds as follows. In the second chapter we briefly summarize existing methods to identify the poor. Afterwards, we briefly present the main axioms which are currently utilised to derive multidimensional distribution sensitive classes of poverty indices. In addition, we introduce the axiom Nonincreasingness under Weak (Strong) Inequality

Decreasing Switch (NIW(S)) which accounts for inequalities between poverty dimensions in an equivalent way as the well-known Transfer Principle does for inequalities within dimensions. The next two chapters build upon this discussion by axiomatically deriving a specific family of poverty measures for cardinal (chapter four) and ordinal (chapter five) data. A specific advantage of this family of poverty measures is that it is so far the only one which accounts for distribution sensitivity within and between (in case of cardinal data) and between (in case of ordinal data) attributes, respectively, while at the same time allowing for an independent relationship between attributes. Chapter six concludes. Throughout the paper, proofs are relegated to the appendix.

2 The Identification Step

In his well-known article from 1976, Sen differentiated two main steps of poverty measurement: i) the identification of the poor, and ii) the aggregation of the identified characteristics of the poor into an overall indicator. This chapter provides a brief description of the identification step but before turning to this issue we will first introduce the denotation we will utilise throughout the paper.

In a population of size *n*, individual *i* possesses a k-row vector of attributes, $x_i \in R_+^k$[3], which is the i^{th} row of a $n \times k$ matrix $X \in K^n$. K^n denotes the set of all $n \times k$ matrices with non-negative entries of real numbers. Let $K = \bigcup_{n \in N} K^n$, whereby *N* is the set of positive integers. The j^{th} column of *X* accordingly denotes the distribution of attribute *j* among the *n* individuals of the population. Thus, the $(i, j)^{th}$ entry of X yields the quantity individual *i* possesses of attribute *j*[4]. Finally, let $z \in Z$; $Z \subset R_+^k$ be the vector of the respective threshold levels chosen for the different attributes.

The first step in the identification of the poor in a multidimensional setting is the identification of the deprived. Donaldson and Weymark (1986) differentiate between a weak and a strong definition of deprivation. According to the *weak definition*, the group of individuals who are deprived with respect to a certain attribute comprises all those who fail to achieve its threshold level. The *strong definition* additionally includes those individuals who reach the respective threshold level. Zheng (1997) claims that the choice of the definition has no empirical implications, while Donaldson and Weymark (1986) demonstrate that the strong definition might have unintentional axiomatic implications. For that matter, we will follow the weak definition by denoting individual *i* deprived with respect to attribute *j* if $x_{ij} < z_j$. For any $X \in K$, let $S_j(X)$ – or simply S_j – denote the set of individuals who are deprived with respect to attribute *j*.

[3] The restriction to positive real numbers is common in poverty measurement since some axioms loose their plausibility in case endowments take non-positive values.
[4] Please note that the quantitative specification of attributes precludes the possibility that variables take a qualitative form.

Once the deprived have been identified, the question is how deprived an individual has to be in order to be called poor. So far, three methods for the identification of the poor can be differentiated, the 'union', the 'intersection' and the 'dual cutoff' method.

The ***'union' method*** denotes an individual poor if his or her achievement level(s) fall short of the respective threshold level(s) in at least one dimension: individual i is poor if $\exists j \in \{1,...,k\}: x_{ij} < z_j$. In this case, any deprived person is automatically poor.

This approach surely accounts for the unique importance of every single poverty dimension since it does not allow for substitution between poor and non-poor attributes, i.e. a shortfall in one dimension cannot be compensated by overachievement(s) in (an)other dimension(s). The obvious disadvantage of this approach is that it is overly inclusive, leading to exaggerated poverty rates.

The ***'intersection' method*** identifies an individual as poor whenever his or her achievement levels fall below the threshold levels of all poverty dimensions: individual i is poor if $x_{ij} < z_j \forall j$.

While this approach obviously identifies the most deprived, it is overly constrictive, leading to minimised poverty rates. For instance, Bourguignon and Chakravarty (1997) criticise that in case longevity and income are two poverty dimensions, an old beggar would not be poor.

The ***'dual cut-off' method*** has been introduced by Alkire and Foster (2009) as a way to combine the two previous methods. While a first cut-off identifies the deprived, a second cut-off defines a minimum number of poverty dimensions, say d, according to which an individual has to be deprived to be poor. Thus, individual i is poor if $x_{ij} < z_j$ for $j \in \{1,...,k\}$ and $\# j \geq d$.

This approach includes both the 'union' as well as the 'intersection' method as special cases where d = 1 and d = k, respectively. One drawback of this approach is that the choice of the second cut-off d is rather arbitrary. However, since the identification of the poor always includes certain arbitrariness not too much emphasis should be placed on this fact. So far, the dual cut-off method seems to be the superior method for counting the poor.

However, when it comes to the aggregation of individual poverty characteristics into an overall poverty measure, the only method that does not waste information on dimension-specific shortfalls is the union method. Still the problem remains that it leads to exaggerated poverty rates. Only but a few papers addressed this issue for the case of ordinal data by introducing different weighting schemes (e.g. Brandolini and D'Alessio 1998; Chakravarty and D'Ambrosio 2006; Jayaraj and Subramanian 2007, 2010; Alkire and Foster 2009; Bossert, Chakravarty and D'Ambrosio 2009).

In this paper we take a different approach. Basically we argue that what ought to be fixed at the aggregation level ought not to be fixed at the identification level. In other words, the exaggeration problem of the union method should not be directly addressed. Instead, we identify an anomaly in the aggregation step which has been induced by an incomplete extension of the Transfer Principle to the multidimensional framework. What we demonstrate is that once this anomaly in the aggregation step is resolved, the exaggeration problem vanishes. It is an argument related to the one Dasgupta and Ray made in 1986 when they demonstrated that distribution issues can either be addressed by a distribution sensitive requirement in the aggregation step or by choosing the "right" poverty line, i.e. choosing the poverty line according to the budget so that all who are poor according to that line are lifted out of poverty. Obviously, both procedures have the same effect; the former, however, seems to be more appropriate. In the same way it seems to be more appropriate to adequately account for distribution sensitivity on the aggregation level instead of choosing the "right" weights or "right" cut-offs.

3 The Aggregation Step

Following the identification of the poor, the subsequent question is how individual poverty characteristics should be aggregated into a single poverty measure. As early as 1976, Amartya Sen introduced a first list of core axioms that reasonable (one-dimensional) poverty indices should satisfy[5]. To the best of our knowledge, so far only four main studies have attempted to generalize and extend the core axioms of the one-dimensional framework to derive a comparable list for the multidimensional framework: Chakravarty, Mukherjee and Ranade (1998); Tsui (2002); Bourguignon and Chakravarty (2003); and Chakravarty and Silber (2008). In the following, we will provide a brief overview of the axioms introduced in these four papers, thereby differentiating between i) core axioms, ii) implied / non-restrictive axioms, and iii) controversial axioms. In addition, we will introduce two versions of a new core axiom, 'Nonincreasingness under Weak (Strong) Inequality Decreasing Switch'.

3.1 The Core Axioms

Core axioms are easily acceptable, independent axioms which are essentially generalizations of axioms proposed in the one-dimensional setting.

Anonymity (AN)[6]: For any $(X;z) \in K \times Z : P(X;z) = P(\Pi X;z)$ where Π is any permutation matrix[7] of appropriate order.

AN states that any characteristic of persons other than the attributes *j* are irrelevant for poverty measurement.

Continuity (CN): For any $z \in Z, P$ is continuous on *K*.

[5] In this paper, we will follow the axiomatic approach for the derivation of indices. Deutsch and Silber (2005) provide a detailed description of the other main methods (i.e. fuzzy set approach, distance function approach, information theory approach) as well as a thorough discussion of the respective advantages and disadvantages.

[6] This axiom is also known as 'Symmetry' (e.g. Tsui 2002).

[7] A permutation matrix is a square (0,1)-matrix of any order that has exactly one '1' entry in each row and each column and '0''s elsewhere.

CN requires P to vary continuously with x_{ij} and is essentially a technical requirement. It precludes oversensitivity of the poverty index, i.e. abrupt changes in P for small changes in X.

Principle of Population (PP)[8]: For any $(X;z) \in K \times Z; m \in N:$[9] $P(X^m;z) = P(X;z)$ where X^m is the m-fold replication of X.

PP ensures that the poverty index depends on the distributions of the attributes j and their shortfalls below z rather than on population size. Thus, by facilitating the transformation of different-sized matrices into one size, PP allows for cross population and cross time comparisons of poverty.

Focus (SF)[10]: For any $(X;z);(Y;z) \in K \times Z:$ if

i) for any h such that $x_{hl} \geq z_l, y_{hl} = x_{hl} + \delta, \delta > 0,$

ii) $y_{il} = x_{il} \forall i \neq h$, $y_{ij} = x_{ij} \forall j \neq l, \forall i,$

then $P(X;z) = P(Y;z)$.

SF demands that giving a person more of an attribute with respect to which this person is not poor will not change the poverty index – even if this person is poor with respect to some other attribute(s).

Subgroup Decomposability (SD): For any $X^1,...,X^m \in K$ and $z \in Z$:

$$P(X^1, X^2,..., X^m; z) = \sum_{i=1}^{m} n_i / n P(X^i; z)$$

with n_i being the population size of subgroup $X^i, i = 1,...,m$ and $\sum_{i=1}^{m} n_i = n$.

SD requires overall poverty to be the population share weighted average of subgroup poverty levels. It thus allows for the decomposition of overall poverty into the poverty levels of population subgroups according to ethnic, spatial or other criteria. SD is a valuable property for policy makers as it allows the calculation of percentage contributions of different subgroups to overall poverty and thus to identify those population subgroups which are most afflicted by poverty.

Factor Decomposability (FD): For any $(X;z) \in K \times Z$:

$$P(X;z) = \sum_{j=1}^{k} a_j P(x_{.j}; z_j)$$

with $a_j > 0$ being the weight attached to attribute j, $j = 1,\dots,k$ and $\sum_{j=1}^{k} a_j = 1$.

FD allows for the decomposition of the poverty index into different attribute combinations. FD and SD together thus allow for the calculation of the contribution of different subgroup-

[8] This axiom is also known as 'Replication Invariance' (e.g. Tsui 2002, Zheng 1997).

[9] N is the set of positive integers.

[10] Bourguignon and Chakravarty (2003) differentiate between a strong and a weak version of the focus axiom. The axiom we introduced as Focus would be the strong version. The definition of the weak version is as follows: For any $(X;z);(Y;z) \in K \times Z$ if for some h $x_{hj} \geq z_j \forall j$: i) for any $l, y_{hl} = x_{hl} + \delta,$ where $\delta > 0,$ ii) $y_{hj} = x_{hj} \forall j \neq l,$ and iii) $y_{ij} = x_{ij} \forall i \neq h, \forall j$, then $P(Y;z) = P(X;z)$. Thus, in contrast to SF, WF requires the poverty index to be independent of the attribute levels of the non-poor persons only. WF follows directly from SF.

attribute combinations to overall poverty. This twofold decomposition of overall poverty improves the targeting of poverty-alleviating policies. An important implication of SD and FD is that their fulfilment requires poverty indices to be additive, i.e. to take the form $P(X;z) = 1/n \sum_{j=1}^{k} a_j \sum_{i \in S_j} P(x_{ij}; z_j)$ (Chakravarty and Silber 2008, p.198).

The next axiom is the result of an argument Amartya Sen made in 1976, requesting poverty indices to be sensitive to inequality among the poor so that, whenever inequality among the poor decreases, poverty should not increase. One well-known partial order, which ranks distributions of attributes by their degrees of inequality, is the so called Pigou-Dalton transfer.

Pigou-Dalton transfer: Matrix X is said to be obtained from matrix Y by a *Pigou-Dalton progressive transfer* of attribute l from one poor individual to another if for some individuals g, h:

i) $y_{gl} < y_{hl} < z_l$,

ii) $x_{gl} = y_{gl} + \theta \le y_{hl}, x_{hl} = y_{hl} - \theta \ge y_{gl}, \theta > 0$

iii) $x_{il} = y_{il} \forall i \ne g,h;\ x_{ij} = y_{ij} \forall j \ne l$ and $\forall i$.

In other words, matrix X is said to be obtained from matrix Y by a Pigou-Dalton progressive transfer of attribute l if X and Y are exactly the same except that the – with respect to attribute l – less deprived poor individual g has θ units less of attribute j in X than in Y, whereas the more deprived poor individual h has θ units more. It is quite reasonable to argue that under such a progressive transfer poverty should not increase. A generalization of this principle is provided by the following axiom.

Transfer Principle (TP): For any $z \in Z$, and X, Y of the same dimension, if $X^P = BY^P$ and B is not a permutation matrix, then $P(X;z) \le P(Y;z)$, where $X^P(Y^P)$ is the attribute matrix of the poor corresponding to $X(Y)$ and $B = (b_{ij})$ is some bistochastic matrix ($b_{ij} \ge 0;\ \sum_i b_{ij} = \sum_j b_{ij} = 1$) of appropriate order.

TP requires that a transformation of the attribute matrix Y^P of the poor in Y into the corresponding matrix X^P by an equalising operation does not increase poverty.

Poverty measures satisfying TP have become known as distribution sensitive poverty measures. They i) distinguish between poverty eliminating, alleviating and redistributing policies, and ii) channel assistance to the poorest individuals first – whereas distribution insensitive measures prioritize the least poor.

TP perfectly accounts for poverty severity in a one-dimensional framework. However, in a multi-dimensional framework it covers only one of two aspects of inequality: TP accounts for inequalities *within* but not *between* poverty dimensions. It therefore only partially covers Sen's request and leads to an anomaly in poverty measurement.

As an illustration for the case of cardinal data, consider the following two situations:

$i = 2, j = 3; z = (4 \quad 4 \quad 4)$

$$Y_1 = \begin{pmatrix} 3 & 4 & 4 \\ 1 & 1 & 4 \end{pmatrix}; \quad X_1 = \begin{pmatrix} 2 & 4 & 4 \\ 2 & 1 & 4 \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} 3 & 4 & 4 \\ 2 & 1 & 4 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 2 & 4 & 4 \\ 3 & 1 & 4 \end{pmatrix}$$

In the first (second) situation, matrix X_1 (X_2) is obtained from matrix Y_1 (Y_2) by a transfer (switch) of attributes providing the poorer individual with an additional unit of an attribute according to which both individuals are deprived. Intuitively, if poverty decreases in the first situation, it should also decrease in the second. However, only in the first situation a decrease in poverty is axiomatically covered by TP.

As an illustration for the case of ordinal data, consider the following situation:

$$i = 2, j = 3; z = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Matrix X is obtained from matrix Y by a switch of attributes reducing (increasing) the number of dimensions in which the poorer (less poor) individual is deprived. Intuitively, if equalising transfers and switches decrease poverty in the situation above, this equalising switch should also decrease poverty. However, this situation is not axiomatically covered.

In order to address the anomalies in accounting for poverty severity, let us first define two kinds of switches.

Weak Inequality Decreasing Switch: Matrix *X* is said to be obtained from matrix *Y* by a *weak inequality decreasing switch* of attribute *l* from one poor individual to another if for some individuals *g*, *h*:

i) $d_g > d_h > 1, d_i = \# x_{ij} < z_j$

ii) $y_{gl} < y_{hl} < z_l$;

iii) $x_{gl} = y_{hl} < z_l$; $x_{hl} = y_{gl} < z_l$;

iv) $x_{ij} = y_{ij} \forall i \neq g, h; \forall j \neq l.$

In other words, a weak inequality decreasing switch provides the poorer individual with a higher amount of an attribute with regard to which both individuals are deprived.

Strong Inequality Decreasing Switch: Matrix *X* is said to be obtained from matrix *Y* by a *strong inequality decreasing switch* of attribute *l* from one poor individual to another if for some individuals *g*, *h*:

i) $d_g > d_h, d_i = \# x_{ij} < z_j$

ii) $y_{gl} < z_l \leq y_{hl}$;

iii) $x_{gl} = y_{hl} \geq z_l$; $x_{hl} = y_{gl} < z_l$;

iv) $x_{ij} = y_{ij} \forall i \neq g, h; \forall j \neq l.$

In other words, a strong inequality decreasing switch reduces (increases) the number of dimensions in which the poorer (less poor) individual is deprived.

In order to account for inequality between dimensions, we introduce the axiom Nonincreasingness under Weak (Strong) Inequality Decreasing Switch for the case of cardinal (ordinal) data.

Nonincreasingness under Weak (Strong) Inequality Decreasing Switch (NIW(S))[11]: For any $(Y;z) \in K \times Z$, if $X \in K$ is obtained from Y by an inequality decreasing switch between two poor individuals, then $P(X;z) \leq P(Y;z)$.

The last but not least two core axioms focus on the relationship between attributes. More precisely, they deal with the poverty implications in case attributes are substitutes or complements[12].

Correlation Increasing Switch[13]: Matrix X is said to be obtained from matrix Y by a *correlation increasing switch* of attribute l from one poor person to another if for some individuals g, h:

i) $y_{hl} < y_{gl} < z_l; \; y_{hm} > y_{gm} < z_m,$

ii) $x_{hl} = y_{gl}, x_{gl} = y_{hl}; \; x_{gm} = y_{gm}, x_{hm} = y_{hm}$

iii) $x_{il} = y_{il}, x_{im} = y_{im} \forall i \neq g, h; \; x_{ij} = y_{ij} \forall j \neq l, m, \forall i.$

That is, under a correlation increasing switch between deprived individuals, the person having a higher amount of one attribute gets a higher amount of (an)other attribute(s) through a rank reversing transfer. Obviously, the effect a correlation increasing switch has on poverty depends on the relationship between attributes and thus implies the following two axioms.

Nondecreasingness under Correlation Increasing Switch (NDC): For any $(Y;z) \in K \times Z$, if $X \in K$ is obtained from Y by a correlation increasing switch of two *substitute* attributes between two poor individuals, then $P(X;z) \geq P(Y;z)$.

Nonincreasingness under Correlation Increasing Switch (NIC)[14]: For any $(Y;z) \in K \times Z$, if $X \in K$ is obtained from Y by a correlation increasing switch of two *complement* attributes between two poor individuals, then $P(X;z) \leq P(Y;z)$.

[11] We are aware that for the special case that attributes are complements, the contrasting axiom Nondecreasingness under Weak (Strong) Inequality Decreasing Switch (NDW(S)) could be formulated. However, we refrain to do so because we especially want to stress the equivalence of NIW and TP in the cardinal case. Whenever TP is satisfied (violated), NIW should be satisfied (violated) as well.

[12] We follow the definition for substitutability, complementarity or independence relationship between attributes commonly utilised in this literature. That is, two attributes are substitutes, complements or independent in case the second cross partial derivative of the poverty measure with respect to these attributes is positive, negative or zero, respectively.

[13] Please note that this is the terminology which Bourguignon and Chakravarty (2003) utilise. Tsui (2002) who first introduced this specific transfer called it a 'basic arrangement-increasing transfer' (Tsui 2002).

[14] Bourguignon and Chakravarty (2003) who introduced NIC stress the relevance of this axiom by referring to the example of education and nutrition. They point out that there seems to be a certain degree of complementarity between these two poverty dimensions. Poor nutrition, especially in the years of early childhood, may lead to persistent health effects which lower educational performance. Thus, it might be possible to actually reduce poverty if better access to education is granted to those children who do not suffer from severe undernutrition. In this case, a correlation increasing switch would lead to a decrease in overall poverty.

It should be noted that due to its different approach to inequality, NIC directly implies the violation of TP as well as NIW.

3.2 The group of implied / non-restrictive axioms

The following group of axioms comprises easily acceptable axioms that are either i) not restrictive or ii) direct implications of the core axioms introduced in the proceeding subchapter.

Monotonicity (MN): For any $(X;z);\ (Y;z) \in K \times Z$ if:
i) for any $h, x_{hl} = y_{hl} + \delta,$ where $y_{hl} < z_l, \delta > 0,$
ii) $x_{il} = y_{il} \forall i \neq h,\ x_{ij} = y_{ij} \forall j \neq l, \forall i,$
then $P(Y;z) \leq P(X;z).$

MN requires the poverty index to not increase if, ceteris paribus, the condition of individual *h* that is poor with respect to attribute *l* improves. MN follows directly from TP and CN.

Nondecreasingness in Subsistence Levels (NS)[15]: For any $X \in K, z \in Z,\ (X;z)$ is non-decreasing in $z_j \forall j.$

NS requires that, ceteris paribus, the population with the higher threshold levels should not have the lower poverty level. NS follows directly from TP and MN.

Non-Poverty Growth (NG): For any $(Y;z) \in K \times Z,$ if *X* is obtained from *Y* by adding a rich person to the population, then $P(X;z) \leq P(Y;z).$

NG requires the poverty index to be nonincreasing in the population size of the non-poor. It follows directly from MN, FC and PP.

Normalization (NM): For any $(X;z) \in K \times Z:\ P(X;z) = 1$ if $x_{ij} = 0 \forall i, j$ and $P(X;z) = 0$ if $x_{ij} \geq z_j \forall i, j.$ Thus, $P(X;z) \in [0,1].$

NM is a cardinality property of a poverty index which simply requires the measure to be equal to zero in case all individuals are non-poor and equal to one in case all individuals are poor. Obviously, this property does not impose very much restriction on poverty indices.

Subgroup Consistency (SC): For any *n* and *k* such that X_1 and Y_1 are $n \times k$ matrices and X_2 and Y_2 are $m \times k$ matrices with[16] $X^T := [X_1^T, X_2^T]$ and $Y^T := [Y_1^T, Y_2^T],$ $P(X;z) > P(Y;z)$ whenever $P(X_1;z) > P(Y_1;z)$ and $P(X_2;z) = P(Y_2;z).$

SC requires the poverty index to not increase in case the poverty degree of a population subgroup decreases. SC follows directly from SD.

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[15] This is a multidimensional extension of the Increasing Poverty Line axiom (Zheng 1997).

[16] X^T is the transpose of matrix *X*. Note that for SI to make sense the values of attributes are required to be positive, a fact we accounted for in defining $x_i \in R_+^k$.

3.3 The group of controversial axioms

This last group comprises well-known axioms which cannot be easily justified and are thus discussed highly controversially.

Poverty Criteria Invariance (PI): Let $z, \tilde{z}$ be such that $z_j \neq \tilde{z}_j$; then $P(X;z) \leq P(Y;z) \Leftrightarrow P(X;\tilde{z}) \leq P(Y;\tilde{z})$ whenever $X(z) = X(\tilde{z})$ and $Y(z) = Y(\tilde{z})$.

Suppose the vector of poverty thresholds is adjusted from z to $\tilde{z}$. If the same group of individuals is identified as poor under the new poverty thresholds, then PI requires the *ordinal ranking* of X and Y to remain unchanged. In other words, a change in thresholds which does not alter the number of the poor should not lead to a significant change in the evaluation of poverty. However, as Tsui (2002) points out, PI precludes possible changes in shortfalls, i.e. from $z - x_{ij}$ to $\tilde{z} - x_{ij}$, which may very well reverse ordinal rankings if differential ethical weights are assigned to shortfalls of attributes.

Translation-Scale Invariance (TI): For any $(X; z) \in K \times Z$: $P(X; z) = P(X + \Gamma; z + t)$, where Γ is any matrix with identical rows $t := (t_1, \ldots t_k)$.

TI requires that adding a constant to the income of each individual as well as to the respective threshold levels does not change the degree of poverty. As in the one-dimensional case TI is the characterisation of absolute poverty indices rather than an actual axiom.

Scale Invariance (SI)[17]: For any $(X;z) \in K \times Z$: $P(X;z) = P(X';z')$ where $X' = X\Lambda$; $z' = \Lambda z$ with Λ being the diagonal matrix $diag(\lambda_1, \ldots, \lambda_k), \lambda_j > 0 \forall j$.

SI ensures that the poverty index is invariant to a scale transformation of attributes and thresholds, i.e. the poverty index does not change when the matrix X and the vector z are multiplied by the same diagonal matrix Λ. In other words, only the relative distance to poverty thresholds matters for poverty measurement. Equivalent to TI, SI is the characterisation of relative poverty indices rather than an actual axiom. In addition, Zheng (1994) shows that it is impossible for distribution sensitive poverty measures to satisfy TI and SI at the same time.

We will now turn to the axiomatic derivation of multidimensional poverty measures.

4 Cardinal Classes of Multidimensional Distribution Sensitive Poverty Measures

To the best of our knowledge, five main classes of multidimensional distribution sensitive poverty measures have been developed so far[18]. All of them have been derived from different combinations of the axioms introduced in the previous chapter. Obviously, something like the 'best measure' does not exist. It is a direct implication of the fact that the fulfilment of one set

[17] Tsui (2002) calls this axiom Ratio-Scale Invariance (RS). Please note that in order for SI to be reasonable, attribute values ought to be positive – a fact we accounted for in defining $x_i \in R_+^k$.

[18] The following is again based on the work of Chakravarty, Mukherjee and Ranade (1998); Tsui (2002); Bourguignon and Chakravarty (2003); and Chakravarty and Silber (2008).

of axioms inevitably leads to a violation of another set of axioms. Thus, the choice any set of axioms will always be context-specific – as will the poverty measure that is derived from it. For instance, Chakravarty, Mukherjee and Ranade (1998, p. 184) show that:

Proposition 1. The only family of poverty measures satisfying CN, FC, SD, FD, SI, MN, TP and NM is of the form $P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j f\left(x_{ij}/z_j\right)$ with $f:[0,\infty] \to R^1$ continuous, non-increasing, convex, $f(0) = 1$ and $f(t) = 0 \forall t \geq 1$.[19] Also, $a_j > 0$ are constants such that $\sum_{j=1}^{k} a_j = 1$. This poverty measure does not satisfy NIW.

As argued in the previous chapter, every class of poverty measures that satisfies TP should also satisfy NIW in order to avoid an anomaly in poverty measurement. Thus, in order to allow for the fulfilment of NIW, we extend proposition 1 in the following way:

Proposition 2. The only family of poverty measures satisfying CN, FC, SD, FD, SI, MN, TP, NM and NIW is of the form $P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \varphi(d_i) f\left(x_{ij}/z_j\right)$ with $f:[0,\infty] \to R^1$ continuous, non-increasing, convex, $f(0) = 1$ and $f(t) = 0 \forall t \geq 1$. Also, $a_j > 0$ are constants such that $\sum_{j=1}^{k} a_j = 1$. Furthermore, $\varphi(d_i)$ is increasing and convex, with $d_i = \# x_{ij} < z_j$, $\varphi(0) = 0$ and $\varphi(k) = 1$.

Please note that the fulfilment of NIW requires the introduction of the weighting function $\varphi(d_i) \in [0,1]$ which counteracts the exaggeration of poverty rates otherwise induced by the union method. Thus by tackling an anomaly in the aggregation step we automatically tackled the exaggeration problem which so far has only been addressed at the identification level.

Three out of the five cardinal classes of poverty measures we present in this paper belong to the special additive family of poverty measures defined in proposition 1: i) the multidimensional Foster-Greer-Thorbecke class of indices, ii) the multidimensional Watts class of indices, and iii) the first multidimensional Chakravarty class of indices. The axiomatically implied additivity of this special class of poverty measures – a direct result of the fulfilment of FD – directly implies independent attributes. These specific measures will therefore inevitably violate NDC and NIC. We will first introduce the poverty measures in their initial form before slightly modifying them by suggesting a matching form of the weighting function $\varphi(d_i)$.[20]

The multidimensional Foster-Greer-Thorbecke (FGT) class of poverty measures:

This class of poverty measures is a multidimensional extension of the Foster-Greer-Thorbecke index from 1984.

[19] Please note that Chakravarty, Mukherjee and Ranade (1998) utilise the weaker form of NM, requiring $f(t) = c \forall t \geq 1$, where $c < 1$ is a constant.

[20] The proofs for all axiomatic properties of the poverty measures presented here are either provided by the respective authors who introduced them (Chakravarty, Mukherjee and Ranade 1998; Tsui 2002; Bourguignon and Chakravarty 2003; and Chakravarty and Silber 2008) or are straightforward extensions of these proofs. They are thus not included in this paper but are available from the author on request.

$$P_{FGT}(X;z) = 1/n \sum_{j=1}^{k} \sum_{i \in S_j} a_j \left(1 - x_{ij}/z_j\right)^{\theta_j}$$
with $a_j > 0; \sum_{j=1}^{k} a_j = 1; \ \theta_j > 1$

This class of poverty measures does not satisfy the axioms NDC and NIC; a direct result of the additivity implied by FD. Though TP is satisfied, NIW is violated. In order to achieve the fulfilment of NIW, we introduce an additional dimension-specific weight which leads to the following modified version:

$$P_{FGT}^{*}(X;z) = 1/n \sum_{j=1}^{k} \sum_{i \in S_j} a_j \left(d_i/k\right)^{\theta} \left(1 - x_{ij}/z_j\right)^{\theta_j}$$
with $a_j > 0; \sum_{j=1}^{k} a_j = 1; \ \theta_j > 1; \ \theta = 1/k \sum_{j=1}^{k} \theta_j \ ; d_i = \# x_{ij} < z_j, \ j = \{1,...,k\}$

The parameter θ_j reflects different perceptions of poverty severity; it can be interpreted as an indicator for poverty aversion. For a given $X, P_{FGT}^{(*)}$ decreases as θ_j increases, that is a smaller θ_j gives greater emphasis to the poorest among the poor – due to our extension with regard to poverty severity within *and* between dimensions. d_i / k is increasing in d_i, i.e. greater emphasis is given to the poorest between dimensions. Please note that the fulfilment of NIW does not lead to a violation of any other axiom.

The first multidimensional Chakravarty class of poverty measures (C_1):

This class of poverty measures is a direct multidimensional extension of the Chakravarty index from 1983.

$$P_{C_1}(X;z) = 1/n \sum_{j=1}^{k} a_j \sum_{i \in S_j} \left[1 - \left(x_{ij}/z_j\right)^{c_j}\right]$$
with $a_j > 0; \sum_{j=1}^{k} a_j = 1; \ c_j \in (0,1)$

The unmodified class of indices satisfies exactly the same axioms as the FGT class of indices. Again, to allow for the fulfilment of NIW, we modify this class by introducing an additional dimension-specific weight:

$$P_{C_1}^{*}(X;z) = 1/n \sum_{j=1}^{k} a_j \sum_{i \in S_j} \left(d_i/k\right)^{1/c} \left[1 - \left(x_{ij}/z_j\right)^{c_j}\right]$$
with $a_j > 0; \sum_{j=1}^{k} a_j = 1; \ c_j \in (0,1); \ c = 1/k \sum_{j=1}^{k} c_j; \ d_i = \# x_{ij} < z_j, \ j = \{1,...,k\}$

As in the case of the FGT class of indices, the parameter c_j can be interpreted as an indicator for poverty aversion. Note that in this case for a given $X, P_{C_1}^{(*)}$ increases as c_j increases, meaning that a larger c_j gives greater emphasis to the poorest among the poor – within and between dimensions. Again d_i / k is increasing in d_i, i.e. greater emphasis is given to the poorest between dimensions. Please note that again the fulfilment of NIW does not lead to a violation of any other axiom.

The multidimensional Watts (W) class of poverty measures:

This class of poverty measures is a direct multidimensional extension of the Watts index from 1968.

$$P_W(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \log\left(z_j / \hat{x}_{ij}\right)$$

with $\hat{x}_{ij} = \min\{z_j, x_{ij}\};\ a_j > 0;\ \sum_{j=1}^{k} a_j = 1$

This class of poverty measures satisfies the same set of axioms as the former classes but in addition satisfies PI. However, NIW is still violated. In order to allow for its fulfilment, we again introduce an additional dimension-specific weight:

$$P_W^*(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \left[1 - \log\left(k/d_i\right)\right] \log\left(z_j / \hat{x}_{ij}\right)$$

with $\hat{x}_{ij} = \min\{z_j, x_{ij}\};\ a_j > 0;\ \sum_{j=1}^{k} a_j = 1;\ \ d_i = \# x_{ij} < z_j,\ j = \{1,\dots,k\}$

Please note that as in the two previous cases the fulfilment of NIW does not lead to a violation of any other axiom.

Obviously, a disadvantage of this class of poverty measures is that its indicator for poverty aversion, the logarithm, is constant across dimensions as it, in difference to θ_j and c_j, does not depend on *j*. However, Chakravarty, Deutsch and Silber (2008) show that this class has the great advantage that it can be decomposed in five elements which allow the identification of the causal factors of poverty: i) the Watts poverty gap ratio; ii) the Bourguignon-Theil index of inequality among the poor; iii) the overall headcount ratio; iv) the weights of the various dimensions; v) a measure of correlation between the various dimensions. In the case of the modified version, a measure of poverty intensity between dimensions is added as a sixth element. This decomposability is obviously a very valuable property for the development of poverty reduction strategies.

We will now turn to the last two classes of multidimensional poverty measures which diverge from the basic additive form of the previous classes. In utilising non-additive aggregation functions, these measures accept the violation of FD in order to allow for a dependent relationship between attributes and thus sensitivity towards correlation increasing switches.

Giving up the restriction of independent relationships between attributes seems to be appealing. Nevertheless, we would like to point out that no method has been provided so far which helps to determine what is a substitute or complement. Thorbecke (2008), for instance, raises concerns over the fact that the relationship between attributes may even change with time, being substitutes in the short-run and complements in the long-run. Also, possible implications for the identification step, especially in case attributes are assumed to be complements, have not been addressed so far. This in mind, we will now present the two non-additive classes of poverty measures, i) the second multidimensional Chakravarty, and ii) the multidimensional Bourguignon-Chakravarty class of poverty measures.

The second multidimensional Chakravarty class of poverty measures (C_2):

This class of poverty measures is a non-additive multidimensional extension of the Chakravarty index from 1983, and has been introduced by Tsui (2002).

$$P_{C_2}(X;z) = 1/n \sum_{i=1}^{n} \left[\prod_{j=1}^{k} \left(z_j / \hat{x}_{ij}\right)^{r_j} - 1\right]$$

with $\hat{x}_{ij} = \min\{z_j, x_{ij}\}$; $r_j \in [0,1]$

A direct result of the non-additivity is the violation of FD. Otherwise this class of poverty measures satisfies the same set of axioms as the (modified) Watts class of poverty indices. Another direct result of the non-additivity is that attributes are no longer independent. In fact, this class of poverty measures assumes substitute attributes, therefore satisfying NDC. As NIW is directly implied by the multiplication of poverty dimensions, no additional dimension-specific weight is needed in order to account for poverty severity between dimensions.

Bourguignon and Chakravarty (2003) criticised this class of poverty measures for its restriction on substitute attributes only. In response, they introduced the following class of poverty measures:

The multidimensional Bourguignon-Chakravarty (BC) class of poverty measures:

$$P_{BC}(X;z) = 1/n \sum_{i=1}^{n} \left[\sum_{j=1}^{k} a_j \left(1 - \hat{x}_{ij}/z_j\right)^{\alpha} \right]^{\delta/\alpha}$$

with $\hat{x}_{ij} = \min\{z_j, x_{ij}\}$; $a_j > 0$; $\sum_{j=1}^{k} a_j = 1$; $\alpha > 1; \delta \geq \alpha \vee \delta \leq \alpha$

Again, in order to allow attributes to be depended, a violation of additivity-requiring FD has to be accepted. Different from the previous class of poverty measures, PI is not satisfied.

As Chakravarty and Silber (2008) point out, this class of indices is less simple than Tsui's multidimensional extension since constant elasticity i) is defined between shortfalls rather than attributes, and ii) does not necessarily equal one. However, the most significant difference is that this class does not restrict attributes to be substitutes in the forefront but instead allows them to be either substitutes ($\delta > \alpha$) or complements ($\delta < \alpha$).

In case attributes are assumed to be substitutes ($\delta > \alpha$), this class of poverty measures satisfies the same set of axioms as Tsui's extension, including the fulfilment of NIW, the only exception being the violation of the (controversial) axiom PI. In case attributes are assumed to be complements ($\delta < \alpha$), however, the axiomatic set changes considerably. A direct implication of this specific relationship between attributes as illustrated by the fulfilment of NIC is the violation of TP and NIW.

The following table summarises the discussions of this chapter by providing an overview of the five classes of poverty measures and the respective set of axioms they satisfy or violate.

Axioms	FGT	FGT*	C_1	$C_1{}^*$	W	W*	C_2	BC
Anonymity (AN)	✓	✓	✓	✓	✓	✓	✓	✓
Continuity (CN)	✓	✓	✓	✓	✓	✓	✓	✓
Principle of Population (PP)	✓	✓	✓	✓	✓	✓	✓	✓
Focus (FC)	✓	✓	✓	✓	✓	✓	✓	✓
Subgroup Decomposability (SD)	✓	✓	✓	✓	✓	✓	✓	✓
Factor Decomposability (FD)	✓	✓	✓	✓	✓	✓	✗	✗

Transfer Principle (TP)	✓	✓	✓	✓	✓	✓	✓	✓✗[21]
Nonincreasingness under Weak Inequality decreasing switch (NIW)	✗	✓	✗	✓	✗	✓	✓	✓✗[23]
Nondecreasingness under Correlation increasing switch (NDC)	✗	✗	✗	✗	✗	✗	✓	✓✗[23]
Nonincreasingness under Correlation increasing switch (NIC)	✗	✗	✗	✗	✗	✗	✗	✗✓[22]
Monotonicity (MN)	✓	✓	✓	✓	✓	✓	✓	✓
Nondecreasingness in Subsistence Levels (NS)	✓	✓	✓	✓	✓	✓	✓	✓
Non-Poverty Growth (NG)	✓	✓	✓	✓	✓	✓	✓	✓
Normalization (NM)	✓	✓	✓	✓	✓[23]	✓[25]	✓	✓
Subgroup Consistency (SC)	✓	✓	✓	✓	✓	✓	✓	✓
Scale Invariance (SI)	✓	✓	✓	✓	✓	✓	✓	✓
Poverty Criteria Invariance (PI)	✗	✗	✗	✗	✓	✓	✓	✗
Translation Scale Invariance (TI)	✗	✗	✗	✗	✗	✗	✗	✗

5 Ordinal Classes of Multidimensional Distribution Sensitive Poverty Measures

As in the cardinal case, the issue of distribution sensitivity between dimensions has not been addressed on the aggregation level. In chapter 3 we introduced property NIS for the case of ordinal data. Equivalent to the cardinal case, we introduce the following proposition:

Proposition 3. The only family of poverty measures satisfying CN, FC, SD, FD, SI, MN, NM and NIS is of the form $P(X;z) = 1/n\sum_{i=1}^{n}\sum_{j=1}^{k} a_j \varphi(d_i)$ with $\varphi(d_i)$ increasing and convex, $d_i = \# x_{ij} < z_j$, $\varphi(0) = 0$ and $\varphi(k) = 1$. Also, $a_j > 0$ are constants such that $\sum_{j=1}^{k} a_j = 1$.

Please note that again by tackling an anomaly in the aggregation step we automatically also solve the exaggeration problem which so far has only been addressed at the identification level.

We will compare this new family of poverty measures with three well-known ordinal classes of poverty measures. Two of them belong to a family of poverty measures originally introduced as a class of multidimensional social exclusion measures by Chakravarty and D'Ambrosio (2006):

$$E^n = 1/n\sum_{i\in S_j} f\left[\sum_{j=1}^{k} a_j\right]$$

with $a_j > 0; \sum_{j=1}^{k} a_j = 1;$ f increasing with a non-decreasing marginal

The following classes of poverty measures are subgroups of this family.

The multidimensional Chakravarty and D'Ambrosio class of poverty measures:

$$P_{CD} = 1/n\sum_{i\in S_j}\left[\sum_{j=1}^{k} a_j\right]^r$$

with $r \geq 1;\ a_j > 0; \sum_{j=1}^{k} a_j = 1$

[21] TP, NIW, NDC are only satisfied in case attributes are substitutes, i.e. for $\delta > \alpha$.
[22] NIC is only satisfied in case attributes are complements, i.e. for $\delta < \alpha$.
[23] Please note that this class of poverty measures is not defined in case all individuals are poor.

Please note that for the interesting cases $r > 1$ attributes are assumed to be substitutes. As in the cardinal case, giving up independence of attributes leads to a violation of Factor Decomposability (FD) but also to distribution sensitivity, i.e. the fulfilment of Nonincreasingness under Strong Inequality Decreasing Switch (NIS).

Jayaraj and Subramanian (2010) demonstrated that in case all attributes are weighted equally, that is $a_j = 1/k \, \forall_j$, we get the following special class of poverty measures:

$$P_{CD} = 1/n \sum_{i \in S_j} \left[\sum_{j=1}^{k} 1/k\right]^r = 1/n\left[(d_1/k)^r + \ldots + (d_n/k)^r\right] \quad (1)$$

Now define $R_j \equiv \{i \in N\{d_i = j\}; j = 1,\ldots,k\}$ with $\# R_j = n_j; j = 1,\ldots,k$ and $H_j = n_j/n$. Then from (1):

$$P_{CD} = \pi_\alpha = \sum_{j=1}^{k} (j/k)^\alpha H_j \text{ with } \alpha \geq 1$$

In the same paper, Jayaraj and Subramanian show that in case α becomes indefinitely large, the resulting poverty measure approximates a sort of "Rawlsian", "maxi-max" measure which measures poverty entirely by the headcount ratio of the most deprived. This is exactly the headcount ratio corresponding to the intersection method of identification, i.e. $\lim_{\alpha \to \infty} \pi_\alpha \equiv H^I$. (Jayaraj and Subramanian (2010, p. 56).

In case $\alpha = 1$ the resulting poverty measure $\pi_1 = \sum_{j=1}^{k} (j/k) H_j$ is exactly the measure which i) Brandolini and D'Alessio (1998) introduce as their Z_1, as well as, pointed out by Jayaraj and Subramanian (2010), the measure which ii) Chakravarty and D'Ambrosio (2006) designate as their E_1, iii) Jayaraj and Subramanian (2005, 2007) utilised for their work on poverty in rural India, iv) Alkire and Foster (2007, 2009) designate as the index M_0 and v) Alkire and Santos (2010) made famous as the Multidimensional Poverty Index (MPI). Please note, however, that in case $\alpha < 2$ the resulting poverty measures are not distribution sensitive, i.e. violate NIS. We will now take a closer look on the Alkire and Foster (2007, 2009) class of poverty measures.

The multidimensional Alkire and Foster class of poverty measures:

$$M_\alpha = 1/nk \sum_{i \in S_j} \sum_{j=1}^{k} w_j^\alpha$$

with $\alpha \geq 0; w_j > 0; \sum_{j=1}^{k} w_j = k$

The Alkire and Foster class of poverty measures is not distribution sensitive, i.e. violates NIS independently of the choice of α. In contrast to the previous class of poverty measures, this class satisfies FD. A case which is especially interesting is the case where $\alpha = 0$ since this directly implies that all attributes have equal weights. As already mentioned, the resulting poverty measure is π_1 which has recently become better known as the MPI:

$$M_0 = 1/n\left[(d_1/k) + \ldots + (d_n/k)\right] = \sum_{j=1}^{k} (j/k) H_j = MPI = \pi_1$$

Another, slightly different class of poverty measures has been introduced by Bossert, Chakravarty and D'Ambrosio in 2009:

The multidimensional Bossert, Chakravarty and D'Ambrosio class of poverty measures:

$$P_{BCD} = \left[1/n \sum_{i \in S_j} \left[\sum_{j=1}^{k} a_j \right]^r \right]^{1/r}$$

with $r \geq 1;\ a_j > 0;\ \sum_{j=1}^{k} a_j = 1$

As in the Chakravarty and D'Ambrosio (2006) case, attributes are assumed to be substitutes for $r > 1$. Besides the resulting distribution sensitivity and violation of FD, the special weighting scheme also leads to a violation of SD. Please note that the weighting is a direct consequence of the so-called "S-convexity" property requiring poverty measures to be inequality averse in individual poverty levels (Bossert, Chakravarty and D'Ambrosio 2009, p. 7-8). Again consider the special case of equal weights, i.e. $a_j = 1/k\ \forall_j$:

$$P_{BCD} = \left[1/n \sum_{i \in S_j} \left[\sum_{j=1}^{k} 1/k \right]^\alpha \right]^{1/\alpha} = \left[1/n \left[(d_1/k)^r + ... + (d_n/k)^\alpha \right] \right]^{1/\alpha} = \left[\sum_{j=1}^{k} (j/k)^\alpha H_j \right]^{1/\alpha} = [\pi_\alpha]^{1/\alpha}$$

We will now turn to our family of poverty measures:

$$P_R = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \varphi(d_i)$$

with $a_j > 0;\ \sum_{j=1}^{k} a_j = 1;\ \varphi$ increasing and convex

Please note that a great advantage of our family of poverty measures is that it is the only one which allows attributes to be independent and still is distribution sensitive, i.e. satisfies NIS.

If we take a look at the special case of equal weights, we will detect a very interesting resemblance to the other classes of poverty measures.

Define $a_j = 1/k\ \forall j.$ Then:

$$P_R = 1/nk \sum_{i \in S_j} \sum_{j=1}^{k} \varphi(d_i) = 1/n \left[(d_1/k)\varphi(d_1) + ... + (d_n/k)\varphi(d_n) \right] \Leftrightarrow$$

$$P_R = \sum_{j=1}^{k} (j/k)\varphi(j) H_j$$

Now consider the following two sub-classes of P_R, the modified Foster Greer Thorbecke and the modified first Chakravarty class of poverty measures for the case of ordinal data:

$P_{FGT}^* = 1/n \sum_{j=1}^{k} \sum_{i \in S_j} a_j (d_i/k)^\theta$ with $\theta \geq 1;\ a_j > 0;\ \sum_{j=1}^{k} a_j = 1$ and

$P_{C_1}^* = 1/n \sum_{j=1}^{k} a_j \sum_{i \in S_j} (d_i/k)^{1/c}$ with $0 < c < 1;\ a_j > 0;\ \sum_{j=1}^{k} a_j = 1$

Again define $a_j = 1/k\ \forall j.$ Then:

$P_{FGT}^* = 1/nk \sum_{j=1}^{k} \sum_{i \in S_j} (d_i/k)^\theta = 1/n \left[(d_1/k)^{\theta+1} + ... + (d_n/k)^{\theta+1} \right] = \sum_{j=1}^{k} (j/k)^{\theta+1} H_j$ and

$$P_{C_1}^* = 1/nk \sum_{j=1}^{k} \sum_{i \in S_j} (d_i/k)^{1/c} = 1/n \left[(d_1/k)^{(1/c)+1} + ... + (d_n/k)^{(1/c)+1} \right] = \sum_{j=1}^{k} (j/k)^{(1/c)+1} H_j$$

That is $P^*_{FGT} = P^*_{C_1} = \pi_\alpha = \sum_{j=1}^{k} \left(j/k\right)^\alpha H_j$ for $\alpha \geq 2$.

The following table summarises the discussions of this chapter by providing an overview of the five classes of poverty measures and the respective set of axioms they satisfy or violate. As we can see, existing classes of poverty measures account for poverty severity by assuming attributes to be substitutes, which implies a violation of FD. It is only the new family of poverty measures which allows for fulfilment of NIS while still allowing for independent attributes and a fulfilment of FD. It is also interesting to note that the MPI violates six axioms, more than any other class or measure.

Axioms	π_0	π_1	π_2	AF[24]	CD	BCD[25]	R
Anonymity (AN)	✓	✓	✓	✓	✓	✓	✓
Continuity (CN)	✗	✓	✓	✓	✓	✓	✓
Principle of Population (PP)	✓	✓	✓	✓	✓	✓	✓
Focus (FC)	✓	✓	✓	✓	✓	✓	✓
Subgroup Decomposability (SD)	✓	✓	✓	✓	✓	✗	✓
Factor Decomposability (FD)	✓	✓	✓	✓	✗	✗	✓
Transfer Principle (TP)	✗	✗	✗	✗	✗	✗	✗
Nonincreasingness under Strong Inequality decreasing switch (NIS)	✗	✗	✓	✗	✓[26]	✓	✓
Nondecreasingness under Correlation increasing switch (NDC)	✗	✗	✗	✗	✗	✗	✗
Nonincreasingness under Correlation increasing switch (NIC)	✗	✗	✗	✗	✗	✗	✗
Monotonicity (MN)	✗	✓	✓	✓	✓	✓	✓
Nondecreasingness in Subsistence Levels (NS)	✓	✓	✓	✓	✓	✓	✓
Non-Poverty Growth (NG)	✓	✓	✓	✓	✓	✓	✓
Normalization (NM)	✓	✓	✓	✓	✓	✓	✓
Subgroup Consistency (SC)	✓	✓	✓	✓	✓	✓	✓
Scale Invariance (SI)	✓	✓	✓	✓	✓	✓	✓
Poverty Criteria Invariance (PI)	✓	✓	✓	✓	✓	✓	✓
Translation Scale Invariance (TI)	✓	✓	✓	✓	✓	✓	✓

6 Conclusion

In this paper, we addressed a well-known problem of multidimensional poverty measurement, i.e. the exaggeration of poverty rates in case the union method is chosen as identification method. We pointed out, that while a couple of papers addressed this problem by either introducing a new identification method or different weighting schemes, their approaches address the problem at the wrong level. Since the exaggeration problem is at first a problem of how to account for poverty severity in a multidimensional framework, it is a problem at the aggregation level rather than the identification level.

We thus claimed that what ought to be fixed at the aggregation level ought not to be fixed at the identification level. In other words, the exaggeration problem ought not to be addressed directly at the identification level – just as the issue of poverty severity in the one-dimensional

[24] For $\alpha > 0$
[25] For $r > 1$
[26] Only for $f(x_i)$ convex

case has not been tackled by the modification of poverty lines at the identification level but by the introduction of the Transfer Principle (TP) at the aggregation level. In particular, we highlighted that the way in which the Transfer Principle has been extended to the multidimensional framework is incomplete and that it is this very fact that causes exaggerated poverty rates in case the union method is applied. A direct consequence of us resolving that issue by introducing the axiom Nonincreasingness under Weak (Strong) Inequality Decreasing Switch (NIW(S)) for the case of cardinal (ordinal) data, is that the exaggeration problem is automatically solved as well.

Finally we utilised NIW(S) alongside with other desirable properties in order to axiomatically derive a family of distribution sensitive poverty measures for the case of cardinal and ordinal data, respectively. We demonstrated that a slight modification of some of the most well-known cardinal poverty measures allows for the fulfilment of NIW and thus the utilisation of the union method. With regard to the ordinal case, we demonstrated that some of the most commonly utilised ordinal poverty measures which have been derived by directly tackling the exaggeration problem at the identification level are indeed special cases of our family of poverty measures. Last but not least we pointed out that a great advantage of the new family of poverty measures is that it is the only one introduced so far that allows for distribution sensitivity (between and within dimensions in the cardinal and between dimensions in the ordinal case) and independent attributes at the same time.

APPENDIX

Proof of Proposition 1. The first part of the proof is provided by Chakravarty, Mukherjee and Ranade (1998, p. 184-185) and is thus not included.

For the second part, claiming that $P(X;z)$ as specified in (1) does not satisfy NIW, suppose that individuals *g* and *h* are both poor, but in comparison to *h*, *g* is deprived in one additional dimension: $y_{gm} < z_m \leq y_{hm}$. Furthermore, $y_{gl} < y_{hl} < z_l$. Let *X* be another matrix derived from *Y* by a *weak inequality decreasing switch*, that is $x_{hl} = y_{gl} < x_{gl} = y_{hl} < z_l$ while $x_{gj} = y_{gj} \wedge x_{hj} = y_{hj} \forall j \neq l$ and $x_{ij} = y_{ij} \forall i \neq g,h; \forall j.$ NIW requires that $P(X;z) < P(Y;z)$. However,

$$P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j f\left(x_{ij}/z_j\right) = 1/n \left[\sum_{i \neq g,h}^{n} \sum_{j=1}^{k} a_j f\left(y_{ij}/z_j\right) + \sum_{i=g,h} \sum_{j \neq l,m}^{k} a_j f\left(y_{ij}/z_j\right)\right] +$$

$$1/n\left[a_l f\left(y_{hl}/z_l\right) + a_m f\left(y_{gm}/z_m\right) + a_l f\left(y_{gl}/z_l\right)\right] = 1/n \left[\sum_{i \neq g,h}^{n} \sum_{j=1}^{k} a_j f\left(y_{ij}/z_j\right) + \sum_{i=g,h} \sum_{j=1}^{k} a_j f\left(y_{ij}/z_j\right)\right]$$

$= P(Y;z)$, contradicting NIW.

Q.E.D.

Proof of Proposition 2. In order for $P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j f\left(x_{ij}/z_j\right)$ to be sensitive to *weak inequality increasing switches,* $P(X;z)$ needs to be extended by some function, say φ, which may not depend on the attributes $j = 1,...,k$ in order to avoid a violation of FD but nevertheless be sensitive to dimension-specific changes. It follows that

$P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \varphi(d_i) f(x_{ij}/z_j)$ with $d_i = \# x_{ij} < z_j$. By MN, $\varphi(d_i)$ is increasing. NIW requires the convexity of $\varphi(d_i)$. The argument is the same as for the convexity of *f* as required by TP (see Chakravarty, Mukherjee and Ranade 1998, p. 184-185). Finally, by NM, $\varphi(0) = 0$ and $\varphi(k) = 1$.

The sufficiency of this proof can be verified by checking that $P(X;z)$ as specified in (2) indeed satisfies NIW. Suppose that individual *g* and *h* are both poor, but in comparison to *h*, *g* is deprived in one additional dimension, say *m*: $y_{gm} < z_m \le y_{hm}$. Furthermore, $y_{gl} < y_{hl} < z_l$. Let *X* be obtained from *Y* by a *weak inequality decreasing switch*, that is $x_{hl} = y_{gl} < x_{gl} = y_{hl} < z_l$ while $x_{ij} = y_{ij} \forall i \ne g,h; \forall j \ne l$. NIW requires that $P(X;z) \le P(Y;z)$.

$$P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \varphi(d_i) f(x_{ij}/z_j) = 1/n \left[\sum_{i\ne g,h}^{n} \sum_{j=1}^{k} a_j \varphi(d_i) f(y_{ij}/z_j)\right] +$$

$$1/n \left[\sum_{i=g,h} \sum_{j=1}^{k} a_j \varphi(d_i) f(y_{ij}/z_j)\right] + 1/n \left[\varphi(d_g)\left[a_l f(y_{hl}/z_l) + a_m f(y_{gm}/z_m)\right] + \varphi(d_h)\left[a_l f(y_{gl}/z_l)\right]\right] \le$$

$$1/n \left[\varphi(d_g)\left[a_l f(y_{gl}/z_l) + a_m f(y_{gm})\right] + \varphi(d_h)\left[a_l f(y_{hl}/z_l)\right]\right] + 1/n \left[\sum_{i=g,h} \sum_{j=1}^{k} a_j \varphi(d_i) f(y_{ij}/z_l)\right] +$$

$$1/n \left[\sum_{i\ne g,h}^{n} \sum_{j=1}^{k} a_j \varphi(d_i) f(y_{ij}/z_j)\right] = P(Y;z)$$

Q.E.D.

Proof of Proposition 3. In order for $P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j f(x_{ij}, z_j)$ to be sensitive to *strong inequality increasing switches,* $P(X;z)$ needs to be extended by some function, say φ, which may not depend on the attributes $j = 1,...,k$ in order to avoid a violation of FD but nevertheless be sensitive to dimension-specific changes. It follows that $P(X;z) = 1/n \sum_{i=1}^{n} \sum_{j=1}^{k} a_j \varphi(d_i) f(x_{ij}, z_j)$ with $d_i = \# x_{ij} < z_j$. By MN, $\varphi(d_i)$ is increasing. NIS requires the convexity of $\varphi(d_i)$. The argument is the same as for the convexity of *f* as required by TP (see Chakravarty, Mukherjee and Ranade 1998, p. 184-185). Finally, by NM, $\varphi(0) = 0$ and $\varphi(k) = 1$.

The sufficiency of this proof can be verified by checking that $P(X;z)$ as specified in (2) indeed satisfies NIS. Suppose that individual *g* and *h* are both poor, but in comparison to *h*, *g* is deprived in at least one additional dimension: $d_g > d_h > 1$. Let *X* be obtained from *Y* by a *strong inequality decreasing switch,* that is $x_{gl} = y_{hl} \ge z_l ; x_{hl} = y_{gl} < z_l$ while $x_{ij} = y_{ij} \forall i \ne g,h; \forall j \ne l$. NIS requires that $P(X;z) \le P(Y;z)$.

$$P(X;z) = 1/n \left|\sum_{i\in S_j / g,h} \sum_{j=1}^{k} a_j \varphi(d_i) + \sum_{j\ne l}^{k} a_j \varphi(d_h + 1) + a_l \varphi(d_h + 1) + \sum_{j\ne l}^{k} a_j \varphi(d_g - 1)\right| \le$$

$$1/n \left|\sum_{i\in S_j / g,h} \sum_{j=1}^{k} a_j \varphi(d_i) + \sum_{j\ne l}^{k} a_j \varphi(d_h) + \sum_{j\ne l}^{k} a_j \varphi(d_g) + a_l \varphi(d_g)\right| = 1/n \sum_{i\in S_j} \sum_{j=1}^{k} a_j \varphi(d_i) = P(Y;z)$$

Q.E.D.

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